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Page 291.

Daniel K. Lazarus.
Sec. 111a.

Latin is a dead language
as dead as dead can be

It killed the ancient Romans
And now its
killing me!

and stated
Problem,

AN ARITHMETIC FOR TEACHERS

AMERICAN TEACHERS COLLEGE SERIES

JOHN A. H. KEITH AND WILLIAM C. BAGLEY, *Editors*

INTRODUCTION TO TEACHING BAGLEY AND KEITH
FUNDAMENTALS IN ENGLISH CROSS
JUNIOR HIGH SCHOOL CURRICULA HINES
AN ARITHMETIC FOR TEACHERS . . ROANTREE AND TAYLOR
PROCEDURES IN HIGH SCHOOL TEACHING WAPLES

Other volumes in preparation

AN ARITHMETIC FOR TEACHERS

BY

WILLIAM F. ROANTREE

NEW YORK TRAINING SCHOOL FOR TEACHERS

AND

MARY S. TAYLOR

NEW YORK TRAINING SCHOOL FOR TEACHERS

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PREFACE

ARITHMETIC has been defined as the science of number and the art of computation. An adult, having learned as a child to add, subtract, multiply, divide, and extract roots, is more or less proficient in the art of computation. The average individual is perhaps not so well equipped to answer questions relating to the science of number, such as: When is a number exactly divisible by 3? Why does one invert the divisor in an example in the division of fractions? What principle is involved in checking by nines an example in multiplication?

The high-school graduate who has chosen teaching as a profession often feels that, having delved into algebra, geometry, and possibly trigonometry, he is taking a backward step when he again approaches the subject of arithmetic, which he studied as a child. He feels that this subject is not meat for the adult. True, arithmetic is a subject studied in childhood. So is a rose a flower that is familiar to a child. Yet children do not become such connoisseurs in the cultivation of roses as to distinguish between the hybrid tea and the hybrid perpetual, the rugosa and the rambler; to understand what fertilizers are needed for encouraging growth, or what remedies for destroying pests; to appreciate the artistic effects of favorites massed in the profusion of a flower exhibit or in

rose gardens of surpassing loveliness. These are the interests of adults who are willing to devote time and thought to the cultivation of this universal favorite.

Just so it is with this child's subject of arithmetic: to one who studies its growth and cultivation, undreamed-of possibilities unfold. Such a student learns that the science of arithmetic consists of a body of definitions and a system of rules and principles which may be as rigidly proved as the propositions of demonstrative geometry; that its historical development forms an intensely interesting chapter in the annals of the past. He finds that he is in company with some of the great ones of the earth; that some of the greatest thinkers of the past pondered the theory of numbers; that some of the prominent scholars of ancient civilizations worked out algorisms¹ for the fundamental operations; that a great scientist² gave time and thought to the multiplication table; that some of the greatest mathematicians of the present day are devoting years of study and research to obtaining more light on the history of arithmetic; and that leading educators are investigating its teaching principles with a view to presenting it more effectively to children.

A teacher's knowledge of this so-called elementary subject should be greater than the academic knowledge which he obtained as a pupil, and which he in turn hopes to have his pupils obtain. An old Scotch adage has it, "He who teaches all he knows teaches more than he knows." As the source of the water supply for a fountain must be higher than the fountain's spray, so the sources of

¹ The art of computing with figures, so called from Al-karismi (spelled also al-Khowārizmī), a prominent Arabic mathematician.

² John Napier, the inventor of logarithms, born 1550.

a teacher's information on a subject should be greater than his output.

The teacher should know something of the historical development of mathematics, for this acquaints him with the stages of mathematical progress. He should understand the logical development of the subject, for this leads to an appreciation of the value of sequence, to the recognition of various types of exercises, and to an understanding of the methodology of subject matter. He should understand something of the utility of the subject — “the human significance of mathematics”; this gives him a zeal for his subject, an assurance in the value of what he presents, and a guide for the formation of good habits in the use of mathematical tools.

The writers feel that a teacher should have in his possession a book that contains the following information :

1. The explanation or definition of every term occurring in the subject of arithmetic that he might be called upon to explain.
2. A proof for every rule and principle occurring in the subject of arithmetic.
3. An explanation of the steps in the performance of any of the operations of arithmetic, a discussion of their logical sequence, and a good form for the written work in the operations.
4. A full explanation of how to solve problems arithmetically, a briefer explanation of various other methods of solution, and an analysis of what constitutes good problem material.
5. A few historical notes on the various topics taught in arithmetic.

6. Approved methods for teaching the different topics in an elementary course in mathematics.
7. An analysis of the different types of lessons to be taught and of the method which is best adapted to each type.

The authors in this volume have attempted to cover all the points mentioned above. In other words, they have been guided by the twofold purpose of providing in one volume (1) a book of information on various items to be taught in arithmetic — the *teacher's knowledge* of the subject; and (2) a discussion of methods of presenting this knowledge to the child — the *methodology of teaching arithmetic*.

For guidance and instruction received over many years, for inspiration and unfailing friendship, and for many of the ideas elaborated in this book, the authors count it their duty and privilege to acknowledge their indebtedness to the late Middlesex A. Bailey, until November, 1923, head of the department of mathematics in the New York Training School for Teachers. They desire to express to the editors their appreciation and thanks for their most helpful criticism and kindly encouragement. They also wish gratefully to acknowledge the help received from many who have preceded them in this field, especially those whose writings have been quoted in this book. Thanks are due, also, to those publishers through whose courtesy quotations from various books and periodicals have been made.

EDITORS' INTRODUCTION

THERE have been two opposing views regarding the preparation of teachers of arithmetic in the elementary school. One view is expressed by the phrase "adequate mastery of subject matter." The other view is that of "methods and devices." Neither view, exclusive of the other, is wholly valid. As a result, many institutions that prepare teachers have required "academic courses" and also "methods courses" in arithmetic. This arrangement, which follows from a recognition of the elements of validity in the opposing views set forth above, has never proved entirely satisfactory. *An Arithmetic for Teachers* is a conscious effort to combine the valid elements of these opposing views into a consistent unitary treatment and to give, at the same time, an historical background which is combined with "margins of knowledge" for the teacher and "insights" into mathematical relations.

The authors of *An Arithmetic for Teachers* have had varied and distinguished teaching experiences. They have also been students of the problems connected with teaching children arithmetic and with preparing teachers to teach children arithmetic. The book has literally been wrought out under the stress and strain of necessity. It will, it is hoped, have a profound and beneficial effect on the teaching of arithmetic in the elementary school.

It is, therefore, with pleasure that the Editors present *An Arithmetic for Teachers*, a professionalized subject-matter textbook, the first of its type, in THE AMERICAN TEACHERS COLLEGE SERIES.

J. A. H. K.

W. C. B.

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AN ARITHMETIC FOR TEACHERS

CHAPTER I

EXPRESSION OF NUMBERS

TEACHER'S KNOWLEDGE

Expressing *how many*. — One of the earliest needs felt by the people of the world was to answer the question *How many?* This gives rise to what is known as *number*, which may be defined, therefore, as follows:

Number is the expression of how many. This definition is, of course, mathematically incomplete, but the concept it presents is broad enough to serve the central purpose in the study of arithmetic.

Some primitive peoples had no names for numbers beyond three or four,¹ any higher number being referred to as "a lot" or "many." To find exactly how many there were in a larger group than could be designated by name, the primitive method was to compare the units in the unknown quantity to the units in some well-known group, as the fingers on one hand — one unit for the thumb, one for the forefinger, one for the middle finger, one for the third finger, and one for the little finger — thus setting up a *one-to-one correspondence* between the individuals of the unknown group and those of the known group. *E.g.*, "as

¹ SMITH, D. E. — *Number Stories of Long Ago*, Chap. I; Ginn and Company.

many as there are fingers on one hand," "as many as a man has fingers and toes," and similar designations were employed to express numbers up to twenty. To express time in years, a notch was cut for each year; time in days was expressed by the number of "suns." To determine a number of prisoners certain Indians would lay aside a small twig for each man as he marched past; when ten men had passed, a large twig would replace the smaller ones; ten large twigs were in turn replaced by a branch. This process of expressing numbers in consecutive order is known as *counting*. It is the means of answering the question *how many*.

Expressing *how much*. — Another early need of the race was to answer the question *How much?* as, How much time? How much length? How much weight? How much value? and so on. When answering the question *How many?* we were dealing with quantity made up of separate things; that is, of units naturally separated from each other. Such quantity is sometimes referred to as *discrete* quantity (*dis*, apart; *cresco*, to grow). But to answer the question *How much?* is less simple because the question pertains not to quantity made up of separate things, but to quantity that is a continuous whole. Such quantity is known as *continuous quantity*; as a distance, a period of time, or the value of something. To answer the question *How much?* one must select some standard as a unit, and then find how many times the quantity to be measured contains the standard unit; in other words, one answers the question *How much?* in terms of *how many* — how much length by how many feet, how much time by how many days, how much value by how many dollars, etc.

It is evident, then, that all measuring requires counting; the difference between counting and measuring is that in counting the unit is at hand, while in measuring the unit must be determined before counting can take place. To illustrate: How many books on the desk? Here the unit, book, is at hand and is specifically named in the question. How long is this line? To answer this question a unit of length must be selected or invented, as the inch or foot. All counting may be thought of as measuring the *how-many-ness* of a quantity, and all measuring is counting the number of times the unit of measure is contained in the quantity measured.

Primitive counting. — As we have seen, the primitive method of counting was to make a one-to-one correspondence of the things in the unknown group with the things in an easily designated group. Certain forest tribes of Brazil counted on their finger joints¹ and hence counted up to three; some tribes counted as many as the number of fingers on one hand, some as many as the fingers on both hands, some as many as the fingers and toes.

The present method of counting. — A more advanced way of counting is to give each number a name. This is only a convenient way of expressing the same idea as was expressed objectively before the number names were invented. This process of counting is so familiar to everybody that only those who know something of its historical development and have studied its principles can appreciate the logical beauty and simplicity of our scheme of naming numbers. That a dozen verbal elements, one, two . . . ten, hundred, thousand, are sufficient for the verbal expres-

¹ CONANT, L. L. — *The Number Concept*, p. 7; The Macmillan Company.

sion of the integers up to 999,999, and that a working knowledge of the system which makes this possible is easily gained by normal children under the age of ten — these facts, when understood, compel us to admire the austere beauty and scientific harmony of our scheme of counting. This scheme clearly deserves detailed study.

Through ten. — Each number is given an independent name: one, two, three, four, five, six, seven, eight, nine, ten. The fact that we have ten fingers no doubt explains why we use ten as a base instead of twelve or sixty and thus have a decimal instead of a duodecimal or a sexagesimal scale.

From ten to a thousand. — The names used in counting from one to ten are employed in counting groups of ten in the same sequence, and combining each group of tens with the names preceding ten. One and ten or eleven (*ain-lif*, two words meaning one and ten), two and ten or twelve (*twa-lif*), three and ten or thirteen, and so on up to nineteen; then two tens or twenty, twenty-one, twenty-two, and so on to twenty-nine; thirty, forty, and so on up to *ninety-nine*. Instead of continuing this plan and using ten tens or “tenty,” a new word *hundred* is used. This word is then repeated with each of the preceding names, then two hundred, and so on up to nine hundred, are in turn repeated with each of the names below a hundred, but *ten hundreds* receives a new name — *thousand*.

Beyond a thousand. — As soon as one has names for counting a thousand single things, he may use these names for counting *groups* of a thousand each. Hence the names beyond a thousand are combinations of one thousand and the nine hundred ninety-nine preceding names, then two

thousand with each of these names, and so on until nine hundred ninety-nine thousand have been thus combined with the names below a thousand; a *thousand thousand* receives the name *million*.

This plan is continued indefinitely. That is, each new name is repeated in succession with the names preceding it until nine hundred ninety-nine of the names have been repeated, after which the next new name is given. The new names beyond a thousand are *million*, *billion*, *trillion*, *quadrillion*, *quintillion*, *sextillion*, *septillion*, *octillion*, *nonillion*, *decillion*, etc.¹

Origin of group names. — The earliest form of counting in groups higher than a thousand was to repeat the name *thousand*. For example, *two thousand thousand thousand* was the earlier name for *two billion*. When the names *million*, *billion*, *trillion*, and so on were first invented by the Italians in the fourteenth century, a *billion* was equivalent to a *million million*, a *trillion* to a *million billion*, etc. This plan is still used by the English and by some of the nations of northern Europe. It was changed by the French in the seventeenth century to our present plan.

It is possible that originally the numbers were called after the names of objects that had been used to represent them. In certain languages the word for *twenty* was the same as the word for *man* (implying that a man has that totality of fingers and toes); ten was expressed by the words *half a man*, five by the word *hand*, etc. These known facts suggest the theory that the names of all small integers may have been derived in a similar fashion.

¹ Names of additional periods are obtained by combining with the termination "illion" the Latin names for eleven, twelve, thirteen, etc.

Writing numbers (notation). — To express numbers in writing by spelling the names in full would be very laborious. Different nations have therefore worked out symbolical schemes for expressing numbers in written form.¹ This process of expressing numbers by means of written symbols is known as *notation*.

Primitive method. — The earliest method of writing numbers was by scoring. This is the transitional step between objective representation and the more improved symbolical schemes. The word *score* means a skar (scar) or scratch. A number was represented by making scratches on a stick, rock, or other convenient surface. The improved method of scoring ( for 13) which enables one to count by fives as well as by ones was undoubtedly suggested by the five fingers on the hand. The word *score* has come to mean *twenty*, suggesting an ancient practice of making a mark to record the completion of counting fingers and toes. The French *quatre-vingts* (*four twenties*, or *eighty*) shows the persistence in modern language of the idea of grouping by twenties.

Hindu notation. — Several plans of notation¹ have been invented by different peoples, but only one of them follows the universal plan of counting by the decimal scale, and because of this fact, it has supplanted nearly all other systems. It was invented by the Hindus, transmitted by them to the Arabs, introduced by the latter into western Europe and called for many years the Arabic notation.² Since historical research disclosed the real

¹ SMITH, D. E. — *Number Stories of Long Ago*, Chapters II, III, IV.

CAJORI, F. — *A History of Mathematics*, pp. 4-8, 11, 53, 64, 69, 72, 88; The Macmillan Company.

² SMITH AND KARPINSKI. — *The Hindu-Arabic Numerals*; Ginn and Company, 1911.

inventors, the system has been given its proper designation.

The Hindu notation makes use of the principle of *position*, or *place value*, a principle not found in other systems. The value of a digit in any given position or order is one tenth of its value in the next position or order to the left. Thus in *11*, the figure in the right-hand place has one-tenth of the value of the one to the left.

The first nine numbers are expressed by as many distinct symbols, *1, 2, 3, 4, 5, 6, 7, 8, 9*. It is not known how early the Hindus invented their system of writing numbers, but it is known to have been used by them as early as 200 B.C. When first used, numbers were expressed by writing the number of each order and expressing the name of the order by the initial letter of that order. Thus two hundred twenty would be *2h 2t*; two hundred seven would be *2h 7u*. Later when it was decided to express the order names by position, the employment of a tenth symbol became necessary to denote the absence of number in any order, and it was then that the *0* was invented. After its invention two hundred twenty was written 220; two hundred seven, 207. The exact date of its invention is unknown, but it was sometime later than 400 A.D. No other system of notation needed such a symbol. Some other systems, such as the Chinese, have now incorporated it, although they continue to use their own symbols for the other digits.

The Hindu scheme, as it has been modified and developed, is as follows.

Each period has its three orders — its units, tens, and hundreds. The order names may also be expressed as *unit-thousands, ten-thousands, hundred-thousands*, etc.

PERIODS														
Trillions			Billions			Millions			Thousands			Units		
7 6 0			6 3 2			6 0 0			0 0 4			0 3 0		
hundreds			hundreds			hundreds			hundreds			hundreds		
tens			tens			tens			tens			tens		
units			units			units			units			units		
ORDERS														

Reading numbers. — The reading of numbers expressed in symbols is not a difficult matter when the plan of writing them is understood. This process of reading numbers is sometimes called *numeration*.

The distinction between numeration and notation is not commonly made. The term *notation* is often used to include both the reading and writing of numbers.

Rules. — In writing and reading numbers a systematic plan of procedure is more helpful than a haphazard one; hence the following rules:

To write numbers less than a thousand. — Beginning at the left write the number of each order in succession by means of a single figure and express the name of the order by position. For example: Three hundred seven, 307. As we write three we think hundred, etc.

To write numbers beyond a thousand. — Beginning at the left, write the number of each period in succession, expressing the period name by position. All periods except the left-hand period must contain three figures. For example:

To write five quintillion sixty-five billion thirty-six

million twenty-five, we write: 5,000,000,065,036,000,025. As we write 5, we think quintillion; the next three zeros, quadrillion; then trillion (writing 000); then billion (writing 065); then million (writing 036); then thousand (writing 000); then twenty-five (writing 025).

To read numbers less than a thousand. — Beginning at the left read the number in the first order and call the name of the order in the singular; read the number of the next order and the order name in the singular, etc. A vacant order is not read; units' order is not called; the order name for tens is read "ty." Thus in reading 564, we say, Five hundred sixty-four.

To read numbers beyond hundreds. — Beginning at the left read the number of the highest period and call the name of the period in the singular, read the number in the next period and call the period name in the singular, etc. A vacant period is not read and units' period is not called. To find the name of the highest period separate the number into periods of three figures each, beginning with units, and call the period names until the highest period is reached.

Thus to read 5689452000000000078, separating by commas we say *units, thousands, millions, billions, trillions, quadrillions, quintillions*, getting 5,689,452,000,000,000,078; then read: Five quintillion, six hundred eighty-nine quadrillion, four hundred fifty-two trillion, seventy-eight.

English plan of writing and reading numbers. — The rules are exactly the same as for numbers beyond a thousand, except that the numbers to be used or written are separated into periods of six figures each. Illustration:

5,689452,000000,000078. Five *trillion* six hundred eighty-nine thousand four hundred fifty-two *billion*, seventy-eight.

Roman notation. — The Romans based their plan of writing numbers on the following plan of counting:¹

By units: *one, two, three, one from five, five, five and one, five and two, five and three, one from ten, ten.*

By tens: *one ten, two tens, three tens, one ten from five tens, five tens, five tens and one ten, five tens and two tens, five tens and three tens, one ten from ten tens, ten tens.*

By hundreds: *one hundred, two hundred, three hundred, one hundred from five hundred, etc.*

The above-named numbers, therefore, take the following written forms:

Units: I, II, III, IV, V, VI, VII, VIII, IX, X.

Tens: X, XX, XXX, XL, L, LX, LXX, LXXX, XC, C.

Hundreds: C, CC, CCC, CD, D, DC, DCC, DCCC, CM, M.

In writing a few thousand as many M's were used; in writing higher multiples of a thousand a bar was employed. For example, 5000 was written $\overline{V}$, and 1,927,426 was written $\overline{MCMXXVII}CDXXVI$.

The Roman notation is not used for numbers above a few thousands; within this limit the rule for writing them is: Write the thousands, then the hundreds, then the tens, then the units, counting by the above plan as you write. Illustration: Write 948. *One hundred from ten hundred, CM; one ten from five tens, — XL; five and three, VIII. Ans. CMXLVIII.*

Other number scales. — The student's knowledge of notation has been acquired early in life. By the time one

¹ BAILEY, M. A. — *A Handy Book in Teaching Arithmetic*, p. 61; M. A. Bailey, Yonkers, N. Y.

enters college, it is so largely a matter of habit and experience that to get a clear understanding of the structure of our system of numbers one must get away from the system to which he is habituated. It will be helpful to consider what the number system would have been had the human race been created with twelve fingers instead of ten, assuming what is highly probable, that the number scale has been determined by the number of fingers. The following table compares our present system with a system based on twelve as a scale or *radix* (root).

DUODECIMAL: RADIX TWELVE		DECIMAL: RADIX TEN	
Name	Symbol	Name	Symbol
One	1	One	1
Two	2	Two	2
Three	3	Three	3
Four	4	Four	4
Five	5	Five	5
Six	6	Six	6
Seven	7	Seven	7
Eight	8	Eight	8
Nine	9	Nine	9
Ten ¹	X	Ten	10
Eleven ¹	B	Ten and one	11
Twelve	10	Ten and two	12
Twelve and one	11	Ten and three	13
Two twelves	20	Two tens and four	24
Eleven twelves	B0	One hundred three tens two	132
Twelve squared	100	One hundred forty-four	144
Twelve squared two twelves ten	12X	One hundred seventy- eight	178

¹ CHRISTOFFERSON, H. C. — "A New Number System"; *School Science and Mathematics*, December, 1924.

It is seen that the principle of Arabic notation may be used with any given radix whatever; that its essential characteristic is its use of the principle of place value and not, as is sometimes asserted, its use of the decimal scale. It is to be observed, furthermore, that the number of symbols, including the zero, must equal the scale number or radix. If the radix *twelve* is used, *ten* and *eleven* would have to be expressed by single symbols; and *twelve* would be written *10*, denoting one group of twelve and no units.

EXERCISES

1. Ask five questions of the *how many* type; of the *how much* type. Translate the latter into the *how many* form. In what respect are the new questions more definite?

2. Suppose a crop of apples has been picked from the trees and now lies in heaps. How could you describe the yield of fruit as answering the question "How many"? As answering the question "How much"? Which method is better? Why? Which method will probably give us a fraction? Why? Mention such a fractional result.

3. What is meant by *one-to-one correspondence* in expressing a number? Illustrate.

4. What is number? What is counting? Explain some primitive ways of counting.

5. Count by hundreds to the next new name after hundred.

6. Count by thousands; by ten-thousands; by hundred-thousands.

7. Beginning with the number that is three less than a million count in succession to the number that is three beyond a million.

8. In the same way count from the third number less than a billion to the third one beyond it, (a) by our plan; (b) by the English plan.

9. How many independent names (*i.e.* not combinations of other names) are used in counting from one to one million?

10. Explain the difference between an order and a period.

11. Name the first six orders; the first six periods; the first six periods by the English plan.

12. How does it happen that we use the decimal scale in counting?

13. At an average rate of two numbers a second how long would it take to count one million? One billion?

14. How much space would a thousand sticks 4 in. by $\frac{1}{8}$ in. by $\frac{1}{8}$ in. occupy if standing in vertical position and closely packed together? A million such sticks? A billion?

15. Read the following:

(a) "Under the Versailles treaty the determination of the sum was placed with the Reparations Commission, but the agreement of 1921 fixed the sum at 132,000,000,000 gold marks."

(b) "We sent abroad some items like these: Flour, 542,874,797 pounds; meat, 572,119,109 pounds; coffee, 39,185,167 pounds; sugar, 106,169,345 pounds; tobacco, 27,449,645 pounds; rice, 25,466,547 pounds; salt, 13,707,276 pounds; pickles, 1,333,210 gallons. . . . A total of 1,616,501,729 pounds of foodstuffs received in France from our entry into the war to the end of November, 1918."

16. (a) A light year is the distance light travels in one year. The velocity of light is 186,324 miles a second. Find the number of miles in a light year and read the number.

(b) The fixed star Betelgeuse is 255 light years distant from the earth. Find the distance in miles and read the number.

17. Express in words:

(a) 1620007892

(b) 500000792145678

(c) 900040789214687

(d) 82100478900000000904

18. In question (a) above what order does 9 occupy? 7? 6?

19. Read the numbers in question 15 by the English plan.

20. Which is the smallest number that would be read differently by the English and French plans?

21. Distinguish between *figure* and *number*.

22. In the example express the number in the first partial product, the second partial product. Do the numbers you have written give the sum of 828?

$$\begin{array}{r} 36 \\ 23 \\ \hline 108 \\ 72 \\ \hline 828 \end{array}$$

23. Write in symbols :

(a) Five quadrillion five billion five.

(b) Four and a half trillion.

(c) Four hundred quintillion sixty-eight trillion forty-seven.

(d) Fifty-seven sextillion forty-seven billion ninety-eight million.

24. At an average rate of two figures a second how long would it take to write the numbers from 1 to 10,000,000?

25. (a) State some of the most important facts in the history of the Arabic plan of writing numbers.

(b) State how it happens that the English and the French have different systems of notation and numeration for numbers above a million.

26. Without the zero how would we write three thousand seven? What principle of writing numbers is made applicable by the use of 0?

27. Find the difference between one billion English and one billion French and read the answer (a) by French plan; (b) by English plan.

28. (a) Show the Roman plan by counting as you write the units and write 9; the tens, and write 90; the hundreds, and write 900.

(b) State the Roman plan for writing numbers from a thousand on and write 67,594.

29. Compare the Roman and the Hindu plans with respect to the scale used, the number of symbols used, the use of the principle of place value, the adaptability to written computation. Try multiplying forty-seven by ninety-three by the Roman plan.

30. Why is there no symbol for zero in the Roman plan?

31. Express in words: MCMLXVIII; MCDLXXXVIICMXXXII, MMCDXCIX.

32. Count up to our 25 by the duodecimal scale; by the trinary scale (radix, 3).

33. What is the expression for 100 in the duodecimal scale? in the trinary scale?

METHODS OF TEACHING

Child's concept of number. — We have considered rather fully the nature and meaning of our present practices with respect to number and its expression; and also,

in a brief and incidental manner, we have outlined the lineal descent of our present number system from its early beginnings some 2500 years ago. It still remains to discuss the principles and methods by which this rich heritage is being passed on to the present generation of children.

Most children learn to count in rote fashion before they enter school; but it is a mistake to suppose that, because a child can say *one, two, three*, . . . he has clear concepts of the numbers he is naming. With some children these words are like any other meaningless jingle; with others the words have been associated with a series of objects, or acts, each name with its object — *four* meaning the *fourth in a series* (the *ordinal* notion); with a few others the words mean *how many in the series*, or *up to this point* (the *cardinal* notion). It goes without saying that the teacher cannot afford to ignore these differences — he must find out what the children know and supplement this knowledge in accordance with what he finds.

One-to-one correspondence. — Exercises involving one-to-one correspondence, without the use of number names, are helpful in clarifying the child's concept of number. To illustrate:

1. See these marks, ///, on the board; take from your envelopes the same number of sticks (or other material).
2. Tell me by clapping your hands how many sticks there are.
3. See how far you can walk taking a step for each stick.
4. I need so many (the teacher holds up four fingers) boys to help me pass the books.

It will require careful planning on the teacher's part to get this kind of work under way, but it goes to the heart

of the matter, injecting meaning into what might otherwise be meaningless drill on words and symbols. Such work, followed by the use of the corresponding number names and symbols, is an obvious application of the maxim, "Go from the concrete to the abstract"—from the idea to its symbolic expression. This work need not be carried beyond *five*, or, at most, *ten*. The same idea is used with higher numbers when children keep score in their games; thus $\diagup \diagdown \diagup \diagdown$ is written for eight.

Rote counting.—Children delight in counting, both with and without objects. The benefits derived from such activity are somewhat analogous to those sought when piano pupils practice the scales. The children acquire a familiarity with the series of natural numbers (integers) which serves as the foundation for learning the fundamental combinations in addition. The sense of the number series as going on and on, in space or in time or both, is strengthened by combining the counting with rhythmic activities such as measuring by steps, spans, or other easily handled units of measure. Much use is also made of counting in number games and rhymes.¹ The resourceful teacher will find many occasions for the incidental use of counting and scoring.

Hindu symbols.—The next step is the expression of number by the Hindu, or Arabic symbols—first the reading and later the writing. In the first year, this work seldom goes beyond 100. These numbers appear on calendars and on the pages of books; the teacher numbers

¹ WENTWORTH-SMITH. — *Work and Play with Numbers*; Ginn and Company.

BAILEY AND GERMANN. — *Number Primer*; American Book Company.

HARRIS-WALDO. — *First Journeys in Numberland*; Scott, Foresman and Company.

spaces on the blackboard so that the children may be sent to any space by number. Care must be exercised to associate the Hindu digits with objects and with other forms of expression. For example :



FIGURE 1.

The procedure in teaching children to write the figures, 1, 2, etc., is identical with that in teaching the writing of a new letter in a penmanship lesson.

In teaching counting, reading, and writing numbers from ten to one hundred, the familiar four-inch kindergarten sticks, and two glass tumblers as containers, constitute the best objective material available. The following exercises illustrate how this material may be used.

Supply the children with from twenty to thirty sticks apiece and with some rubber bands. Have them make bundles of ten, leaving the other sticks loose.

1. *Proceed from the objects to the number names.* Hold up your bundles in your left hand and the other sticks in your right hand. How many bundles have you? Two tens are called *twenty*. Have you more than twenty sticks? How many more, Helen? Twenty and three more are called *twenty-three*.

2. *Proceed from the number names to the objects.* Who has twenty-six sticks? Put yours in the tumblers, Mary — the twenty in the left-hand glass and the six in the right-hand glass.

3. *Proceed from the objects to the written symbols.* The teacher should have the glasses in plain sight of all the children, and, near by, ten cards, two inches square, bearing the symbols 0, 1, 2, . . . 9.

John has just placed twenty-four sticks in the glasses. Will Susan pick out the number card which tells how many tens there are? Stand the card before the *tens* glass. Now pick out the number card for the other glass and place it properly. The cards say *twenty-four*. Let us all write twenty-four on the board (or on paper).

4. *Proceed from the names to the symbols, without the objects.* Have the sticks collected, but ask the children each to remember how many sticks he had.

How many sticks did you have, John? Stand the cards against the glasses to show twenty-five. Let us all write twenty-five.

5. *Proceed from the written symbols to the names, using objects if difficulties arise.*

(The details are left to the student.)

It is obvious that the above exercises could not all be covered in a single class period; also that there are many different ways of working out the details. The above work is given to indicate the possibilities; the essential feature of the development is that of taking in turn each of the several representations of number as the starting point (stimulus) and going to the other representations (reaction to stimulus). In this manner the desired motor-visual-auditory complex is established in consciousness.

The methods employed in the second year are essentially the same as those in the first year. However, it is not necessary to use objects so much; sometimes the mere allusion to the objective material will serve quite as well as the objects themselves in making clear a difficult point. It should be borne in mind that the sole purpose of the perceptual aids is to establish clear concepts.

Teaching Roman notation. — Roman numerals through XII are sometimes taught in the second year, presumably because of their use on the clock face. Inasmuch as the majority of modern clocks and watches have the

Hindu numerals the above reason lacks force ; moreover, it is doubtful if children would recognize the numerals on the clock face, placed, as they are, at various angles from the vertical position. Do we not all of us see the *positions* of the hour marks rather than the numerals themselves? For these reasons many courses of study defer the teaching of Roman numerals until the children are old enough to learn them in systematic fashion. Even if the system is to be taught piecemeal, the plan of counting on page 10 will be found helpful. The children's knowledge of scoring can be utilized ; the expression *////* easily becomes *IIII*, one of the recognized forms in the Roman system ; the forms *IV*, *VI*, etc., require an understanding of the counting scheme referred to above.

The essential idea of the method (p. 10) is to associate the scheme of writing numbers with a special way of counting. The essential steps in the teaching are : (1) to have the children count in harmony with the plan of writing, (2) to give practice in translating *four* into *one from five*, *six* into *five and one*, etc., (3) to have the pupils read the written numbers (*IX* would be read *one from ten*, *nine*), (4) to have numbers written (*seven* would be thought of as *five and two*, then written *VII*).

Teaching large numbers. — As children advance in the grades larger and larger numbers are encountered. It is good method to introduce very large numbers in the situations in which they occur. If the teacher hands the pupil a news item of interest and asks him to read it, he will desire to gain the ability to read the numbers involved. Arithmetical curiosities like the following lead to large numbers suitable for practice in the upper grades.

$$1 \times 9 + 2 = 11$$

$$12 \times 9 + 3 = 111$$

$$123 \times 9 + 4 = 1111$$

$$\vdots$$

$$12345678 \times 9 + 9 = 111111111$$

$$12345679 \times 8 = 98765432$$

$$12345679 \times 9 = 111111111$$

$$1 \times 8 + 1 = 9$$

$$12 \times 8 + 2 = 98$$

$$123 \times 8 + 3 = 987$$

$$\vdots$$

$$123456789 \times 8 + 9 = 987654321$$

$$9 \times 9 + 7 = 88$$

$$98 \times 9 + 6 = 888$$

$$987 \times 9 + 5 = 8888$$

$$\vdots$$

$$98765432 \times 9 + 0 = 888888888$$

$$123456789 \times 18 = 222222222$$

$$123456789 \times 27 = 333333333$$

$$123456789 \times 36 = 444444444$$

$$\vdots$$

$$123456789 \times 81 = 999999999$$

But the mere mechanical ability to read large numbers is of little value unless the reader has some sense of their magnitude. A child is likely to think of an eight-figure number as about twice as large as a four-figure number instead of around ten thousand times as large. Occasional exercises in roughly comparing numbers of different sizes should be given. In the upper grades the graphic methods which are coming into vogue in the schools will aid in the interpretation of large numbers. Also occasional problems of this type: If our last year's wheat crop had been loaded on freight cars and these cars placed in one long train beginning at New York City and stretching westward, how far would the train reach? Such a problem, if round numbers were used, would not be too tedious; and its solution would serve to convey a much more vivid impression of number and quantity than would any merely abstract discussion of the *periods* and their relations.

EXERCISES

1. State the steps in teaching to count from 10 to 20.
2. (a) Show the steps in teaching pupils to write numbers from 10 to 100.
(b) Give devices which may be used in having pupils take these steps.
3. (a) Up to what point in the number scale would you teach writing numbers by centering the attention upon the orders?
(b) How would you proceed in teaching the writing of larger numbers?
4. Give steps to be taken by pupils in learning to write numbers from a thousand to a billion.
5. State the steps to be taken in reading numbers of more than one period.
6. Explain your method of teaching Roman numerals through XII.
7. A child writes IL for 49. Explain the nature of the mistake and make the correction.

CHAPTER II

ADDITION

TEACHER'S KNOWLEDGE

HISTORICAL DEVELOPMENT

Addition a natural outgrowth of counting. — We have caught a glimpse of the struggles that the peoples of the world had in perfecting systems of counting, and of reading and writing numbers. Different systems were in use among different nations for centuries before *one* system was found of sufficient merit to be adopted almost universally.

The next step in advance was a comparatively simple one, and a natural outgrowth of the process of counting. It was as though the road of progress, having led over steep and difficult ascents, beset with formidable obstacles, had suddenly smoothed out into a little stretch of easy climbing, free from hazards and perils.

Early methods of adding. — This new and somewhat easy process, which in time came to be called *addition*, consisted in combining separate numbers into a whole. The number in the whole was easily found by counting. Various mechanical aids were used to assist in the process. One of the most common of these was the *abacus*. (This word is probably from a Semitic word meaning *dust*.) Different forms of the abacus were in use among different nations. One form consisted of lines drawn in the dust

to represent the orders. Upon these lines pebbles (*calculi*) were placed to represent the number of units, tens, hundreds, etc. Another form consisted of ruled lines on a table on which discs or counters were placed instead of pebbles. In some forms pebbles were moved in grooves; in others, pebbles or beads were strung on a wire. The Chinese and Japanese still employ the abacus — which they call a *suan pan* — in calculating. In some forms the lines were vertical, in others, horizontal. When counters were used, they were actually *carried* to the next higher order when the sum in any order gave an equivalent of the higher order.

The underlying principle in use in all forms of the abacus was the same. Each number to be added in any order, as units', was represented by placing that number of pebbles or counters on the line representing that order and carrying *one* to the next line when the sum equaled or exceeded ten. The final result

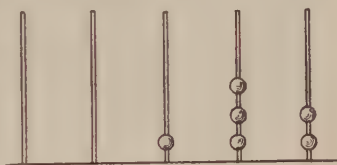


FIGURE 2

was recorded on the abacus. Thus, to combine forty-seven and eighty-five, the seven counters and the five counters were counted off; one was carried to the tens' line and two left on units'; four and eight were then counted off on tens' line, one was carried to hundreds' and two left on tens'; this, with the one carried from units', made a total of three on tens' line. The final result appeared as in Fig. 2. Figure 3 shows an improved form of abacus which uses groups of five on the upper portion.

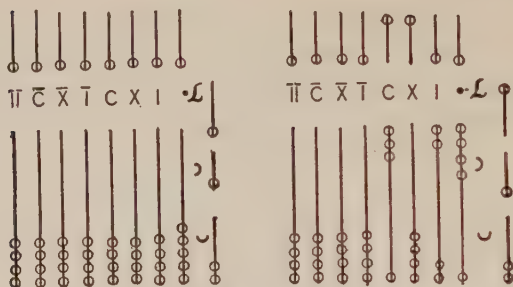


FIGURE 3

Improved methods. — After the Hindu system of notation came into general use improvements were made in adding. Instead of using lines to represent orders it was easy to write the numbers of each order in a column of their own; instead of representing these numbers by pebbles or counters or beads, the numbers were added by taking advantage of the combinations, that is, adding $\frac{7}{2}$ always gave 9 for a result whether the numbers occurred in units', tens', or hundreds' place. The symbols were written in the sand and erased after the addition had been performed, as shown at the right:

Although the process of addition is a comparatively easy one, few ever become really proficient in performing it. Perhaps it is for this reason that the world has again turned to the use of mechanical aids. Instead of an abacus we now employ adding machines; instead of moving beads on a wire we touch keys on a keyboard and the machine does the rest.

A modern standard test in addition. — If the reader wishes to test his own proficiency in addition, he may

try the following examples taken from a well-known standard test :¹

7	9	4	7	2	9	6	7	7	8	9	4	3	2
5	2	5	1	9	6	2	1	8	0	5	3	1	1
4	4	8	9	4	2	6	5	5	7	3	7	7	6
2	8	1	4	8	4	8	1	4	1	4	7	6	6
6	2	4	3	5	7	0	4	1	8	6	0	9	1
0	7	8	2	11	1	4	6	8	5	2	2	6	8
5	5	5	8	5	3	3	5	2	1	3	9	3	6
1	3	1	5	2	9	7	3	1	3	9	5	4	9
8	6	3	2	4	2	1	3	3	7	2	6	5	7
3	1	9	7	3	3	6	7	9	4	2	3	4	5
2	4	6	7	6	8	0	6	8	9	8	4	2	2
9	8	3	1	7	5	6	1	4	4	5	8	9	2
9	8	5	9	6	5	6	7	5	4	6	8	9	4
<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>

Work as many of these examples as you can in two minutes. The median score (p. 302) of rights for the eighth grade is 5.8. A teacher should be able to work from 10 to 12 correctly in the length of time allowed for the test. Where do you stand?

Let us turn to a technical and professional discussion of this operation, so easy to understand but so difficult to perform.

DEVELOPMENT OF SUBJECT MATTER

Definitions. — *Addition* is the process of finding the number of things in a whole from the number of things in each of the several groups that make up the whole. Each of the numbers to be added is called an *addend*. The result obtained by adding is called the *sum*.

¹ Cleveland Survey Test — Set J.; The Public School Publishing Company, Bloomington, Ill.

Only those things which are exactly alike, or for the purpose in view are thought of as being alike, may be combined into a single group. Thus, we may combine a group of 3 apples and another group of 2 apples into a single group of 5 apples; we may combine 3 maples and 2 birches into a group of 5 trees.

The sum of two numbers may be expressed by using the sign (+), which is read *plus*. $5 + 3$ means the sum of 5 and 3; it is read *five plus three*.

Methods of performing addition. — The *fundamental method* of performing addition is by counting. This is necessarily true since counting is the only process which logically precedes addition. The problem may involve the addition of tangible objects ready at hand, in which case it is necessary merely to do the counting; or it may involve the addition of intangible or remote objects, in which case tangible objects are taken to represent the objects of thought.

To illustrate: John is eight years old; how old will he be four years hence? Represent the years of his present age and the years to be added by any convenient objects. John is one, two, three, four, five, six, seven, eight years old (making a mark at each count);

//////////

in one, two, three, four years more (making a mark at each count) he will be one, two, three, . . . eleven, twelve years old.

As has been stated, various forms of tangible objects have been employed by different peoples of the world: twigs, pebbles, knots in a string, beads on a wire, notches in a stick, etc. The fundamental method by counting is shortened (1) by adding together all the units separately, then all the tens, then the hundreds, etc., and (2) by

Two may be increased by each of the nine digits, but logically considered $1 + 2$ is the same combination as $2 + 1$ which occurs in the 1's; hence $2 + 2, 2 + 3$, up to $2 + 9$, or 8 new combinations arise. Then we have $3 + 3$ up to $3 + 9$; then $4 + 4$ up to $4 + 9$, etc., up to $9 + 9$. This gives forty-five combinations in all. Including the zero combinations, $0 + 0, 0 + 1$ up to $0 + 9$, there are fifty-five combinations.

Since the visual image of $\frac{2}{1}$ is not the same as $\frac{1}{2}$, we may say that from the psychological point of view we must add to the above list the inverses of all the combinations in which the addends are not alike, as $5 + 3$, which is the inverse of $3 + 5$. These 100 combinations appear below, arranged in tables according to a constant addend.

0 0 0 0 0 0 0 0 0 0	1 1 1 1 1 1 1 1 1 1
0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9
2 2 2 2 2 2 2 2 2 2	3 3 3 3 3 3 3 3 3 3
0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9
4 4 4 4 4 4 4 4 4 4	5 5 5 5 5 5 5 5 5 5
0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9
6 6 6 6 6 6 6 6 6 6	7 7 7 7 7 7 7 7 7 7
0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9
8 8 8 8 8 8 8 8 8 8	9 9 9 9 9 9 9 9 9 9
0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9

The memorizing of these combinations is the foundation for all skillful work in addition. For if the result of any combination as $6 + 3$ is thoroughly known, the corre-

sponding decade series $16 + 3$, $26 + 3$, etc. is easily learned. And when all the combinations and decades are known, column addition is easily performed. Thus to add skillfully it is necessary to call at once the result of 2 and 4, a combination; to this result, 6, 7 must be added; $7 + 6$ is also a combination; this result, 13, must be increased by 6. $13 + 6$ is a decade. Hence time spent in memorizing the results of the one hundred combinations is time saved for all work that is to follow in addition. The logical steps in learning the combinations in addition are:

1. Obtain the results by counting objects.
2. Memorize the results by such associations as will help to fix them in mind, by sufficient repetitions, by intensive drills on the more difficult combinations, and by frequent and varied applications.

The decades. — Associated with each combination in addition is a series of combinations of numbers of two orders, all ending in the same digit, increased by a number of one order. Thus with $5 + 3$ are associated $15 + 3$, $25 + 3$, and so on up to $95 + 3$. In such a series each number of two orders is ten (decem) more than the preceding number; hence the term *decade*. With each of the ninety numbers of two orders may be associated any one of the nine numbers of one order, giving rise to 810 decade combinations.

The logical steps in mastering the decades are:

1. Determine what the tens of the answer will be by considering whether the sum of the units is ten or more. If it is ten or more, the tens of the answer will be one more than the tens' figure of the number increased; if not,

the tens' figure will remain the same. Thus, in $25 + 3$ the answer will be twenty-something, but in $29 + 3$ the answer will be thirty-something.

2. Determine what the units of the answer will be by associating the decade with the corresponding combination. Thus, in $29 + 3$, the units of the answer will be 2, since $9 + 3$, the combination with which it is associated, equals 12, a number which ends in 2.

3. Practice calling results to acquire skill.

The forty-four series. — A valuable drill employing both combinations and decades consists in counting by any digit beginning with any digit. From an examination of the possible series thus obtained it will be seen that there are two series of 2's, three of 3's, etc., or 44 series in all, as follows:

1, 3, 5, 7, 9, 11, 13.....	By 2, beginning with 1
2, 4, 6, 8, 10, 12.....	By 2, beginning with 2
1, 4, 7, 10, 13, 16.....	By 3, beginning with 1
2, 5, 8, 11, 14, 17.....	By 3, beginning with 2
etc. up to	
9, 18, 27.....	By 9, beginning with 9.

The series of 1, beginning with 1, *i.e.*, 1, 2, 3, 4, 5, etc., might be added as the 45th counting series. But this is so important a series as to have a place and a name of its own — the *natural number series* — and hence is not included in the so-called counting series.

Column addition. — In column addition examples of two addends only in which no column total exceeds nine no special difficulty is presented, since it is as easy to add tens as to add units. Each column is added separately and the sum is written directly under the column

as in example (a). When there are two or more addends, in single or multiple column addition, the sum of the first two must be held in mind and increased by the next addend, as in example (b).

(a)		(b)			
				53	
35	$5u + 2u = 7u$			21	$4u + 1u = 5u; 5u + 3u = 8u$
32	$3t + 3t = 6t$			14	$1t + 2t = 3t; 3t + 5t = 8t$
<u>67</u>				88	

When the sum of any column exceeds 9, the example requires so-called carrying. In mastering column addition with carrying two things are necessary:

1. An understanding of every step in the process which is made clear through the *complete explanation*.
2. The mechanical mastery of the process through use of the *working explanation*, which consists of speaking as little as possible while one works.

Illustration: 236 *Complete explanation:* The sum
 358 of the units is 21 units, which equals
 467 2 tens and 1 unit; write the 1 in
 1061 units' order and add the 2 tens
 with the tens. The sum of the tens is 16 tens, which equals 1 hundred and 6 tens; write the 6 in tens' order, etc.
Working explanation: 14, 21; 5, 10, 16; 3, 6, 10. To form the habit of saying "7 and 8 are 15 and 6 are 21 put down the 1 and carry the 2" etc. is to interfere seriously with an efficient performance of the process. The habit of speaking results only as each figure is added should be a definite part of all practice work in column addition.¹

¹ CONRAD, H. E., and ARPS, G. F. — "An Experimental Study of Economical Learning," *American Journal of Psychology*, p. 507, 1916. Conrad and Arps in-

Mental addition. — Proficiency in mental work is perhaps best acquired by the method of beginning with the highest order first. Thus, in adding 46 and 39, increase 46 by 30 and this result by 9. The working explanation is: 46, 76, 85. In adding 234 and 328 the pupil says: 234, 534, 554, 562. The same procedure may be employed for any number of addends, by finding the sum of the first two, then of that result and a third, etc. People who in column addition add two columns at a time use this method.

Proofs or checks. — Accuracy in column addition requires the employment of some device which provides for going over the work a second time. No human machine is perfect enough to perform a piece of work without possible errors, and the only way to provide against such in the finished result is to check the work.

The common proof consists in adding each column a second time, using the opposite direction from that employed in the first addition. A helpful form of arranging the work for the common proof is shown below :

769	33✓	32
387	36✓	
482	26✓	
569		
456		
2663		

The units are added from the bottom up and the sum, 33, written at the right ; if in adding down a different sum,

investigated the effects upon rate and accuracy of training pupils to use an abbreviated form of "inner speech." The same amount of practice was given two groups, one group using a very brief working explanation and the other a very full one. The brief form showed marked superiority both in speed and accuracy, and also less variability within the group.

as 32, is obtained, this result is written to the right of the first; a third addition is then necessary to find the true result. A check at the right indicates that this result has been found twice, and is presumably the correct result. The result for each succeeding column is placed under that of the preceding and the same procedure is followed for the checking of each column.

Other methods of checking, such as the *civil service method*, the *method of partial sums*, and the *casting out of 7's, 9's, or 11's* may be employed, but no one of these is so useful, so convenient, and so concise as the common proof. The number to be carried to the next column is always in sight in the common proof, and the work, if interrupted, can easily be resumed at the point of interruption. The *civil service* and *partial sums* methods are illustrated below :

CIVIL SERVICE METHOD	PARTIAL SUMS
(a)	(b)
769	769
387	387 1156
482	482
569	569
456	456 1507
<u>33</u>	<u>2663</u> <u>2663</u>
33	
<u>23</u>	
2663	

In (a) each column total is written so that its right-hand figure stands beneath the column giving it; then these partial sums are added. In (b) the example is separated into parts, the sum is found for each part, and these partial sums are then added; this sum should equal the original sum, obtained by the first addition.

To prove by casting out 7's, 9's, or 11's, find the excess of 7's, 9's, or 11's in each addend; add these excesses; the excess of this sum should equal the excess of the original sum. The excess of 7's, 9's, or 11's of a number is the remainder found when the number is divided by 7, 9, or 11, respectively.

EXCESS OF 7's	EXCESS OF 9's	EXCESS OF 11's
769 6	4	10
387 2	0	2
482 6	5	9
569 2	2	8
456 1	6	5
<u>2663</u> $\bar{3}\checkmark$	<u>8</u> $\checkmark$	<u>1</u> $\checkmark$

In Pacioli's arithmetic, published in 1494,¹ examples in the fundamental operations are proved by casting out 7's. This is a laborious process as the excess must be found by dividing each addend by 7.

The excess of 9's or 11's may be found in an easier way.

Let us discover a rule for finding the excess of 9's in a number. *Inductive Procedure*:² Take 258, 974, and 324 and divide each number by 9.

$$\begin{array}{r} 9)258 \\ \underline{28} \end{array} \text{ with 6}$$

$$\begin{array}{r} 9)974 \\ \underline{108} \end{array} \text{ w 2}$$

$$\begin{array}{r} 9)324 \\ \underline{36} \end{array} \text{ w 0}$$

Now let us add the digits of each number; then add the digits of the result:

$$\begin{array}{l} 2 + 5 + 8 = 15 \\ 1 + 5 = 6 \end{array}$$

$$\begin{array}{l} 9 + 7 + 4 = 20 \\ 2 + 0 = 2 \end{array}$$

$$\begin{array}{l} 3 + 2 + 4 = 9 \\ 9 \text{ may be counted as } 0 \end{array}$$

¹ SMITH, D. E. — *History of Mathematics*, p. 252.

² The inductive method of discovering a general rule consists in examining several cases, noting a truth common to all these cases, and assuming it to be probably true for all similar cases. In the deductive method of proof we base our statements upon previously accepted truths. For a discussion of these two methods see Chap. XXI.

In each of the above examples it will be seen the final result is the same as the excess of 9's in the original number. In 324 we may say 9 is contained 35 times with 9 remaining, or 36 times with 0 remaining; it is evident that we may use either 9 or 0 as our remainder. Hence the probable rule: To find the excess of 9's of a number add its digits, add the digits of this result, and so proceed until a number of one order is obtained.

Or it may be stated as a principle, thus: The excess of 9's in a number equals the excess of 9's in the sum of its digits.

Inductive discovery of rule for proving addition by nines. — Take several addition examples that have been checked by the common method. Find the excess of 9's in each addend and place it at the right of the addend. We see that the excess of 9's in 2179 is 1 and the excess of 9's in $4 + 8 + 7$ is 1; the excess of 9's in 1400 is 5 and the excess of 9's in $8 + 6 + 9$ is 5; etc.

526	4	629	8
728	8	537	6
925	7	234	9
$\overline{2179}$	$\overline{1}$	$\overline{1400}$	$\overline{5}$

From the examples tried it seems probable that the excess of 9's in the sum of the excesses of the addends equals the excess of 9's in the sum of the addends themselves. Hence the rule: To check addition by 9's, add the excesses of 9's of the addends; the excess of 9's of this result should equal the excess of 9's of the original sum.

The rule for finding the excess of 11's of a number may be proved by a procedure similar to that used above for the 9's: To find the excess of 11's of a number sub-

tract the sum of the digits in the even orders from the sum of the digits in the odd orders, and increase or decrease this result by that multiple of 11 which will give a positive number less than 11. Illustration: In 3568, the excess is $(8 + 5) - (6 + 3)$ or 4. In 5386, the excess is $(6 + 3) - (8 + 5) + 11$ or 7.

EXERCISES

1. Call the sums of the following, recording your time in seconds:

<u>7</u>	<u>2</u>	<u>5</u>	<u>3</u>	<u>1</u>	<u>1</u>	<u>9</u>	<u>6</u>	<u>3</u>	<u>0</u>	<u>1</u>	<u>0</u>	<u>8</u>	<u>5</u>	<u>0</u>	<u>8</u>
<u>9</u>	<u>1</u>	<u>8</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>6</u>	<u>5</u>	<u>2</u>	<u>3</u>	<u>7</u>	<u>5</u>	<u>8</u>	<u>9</u>	<u>0</u>	<u>7</u>
<u>0</u>	<u>7</u>	<u>5</u>	<u>4</u>	<u>3</u>	<u>8</u>	<u>3</u>	<u>8</u>	<u>6</u>	<u>5</u>	<u>7</u>	<u>0</u>	<u>6</u>	<u>0</u>	<u>0</u>	<u>4</u>
<u>1</u>	<u>8</u>	<u>7</u>	<u>9</u>	<u>4</u>	<u>9</u>	<u>3</u>	<u>6</u>	<u>6</u>	<u>6</u>	<u>7</u>	<u>2</u>	<u>4</u>	<u>4</u>	<u>6</u>	<u>5</u>
<u>1</u>	<u>1</u>	<u>7</u>	<u>2</u>	<u>9</u>	<u>5</u>	<u>2</u>	<u>4</u>	<u>6</u>	<u>1</u>	<u>5</u>	<u>2</u>	<u>8</u>	<u>4</u>	<u>2</u>	<u>9</u>
<u>3</u>	<u>0</u>	<u>6</u>	<u>2</u>	<u>7</u>	<u>5</u>	<u>3</u>	<u>8</u>	<u>7</u>	<u>6</u>	<u>1</u>	<u>4</u>	<u>5</u>	<u>4</u>	<u>0</u>	<u>8</u>
<u>8</u>	<u>3</u>	<u>9</u>	<u>2</u>	<u>8</u>	<u>0</u>	<u>7</u>	<u>1</u>	<u>4</u>	<u>7</u>	<u>6</u>	<u>6</u>	<u>0</u>	<u>6</u>	<u>1</u>	<u>3</u>
<u>0</u>	<u>9</u>	<u>5</u>	<u>5</u>	<u>4</u>	<u>9</u>	<u>5</u>	<u>4</u>	<u>3</u>	<u>5</u>	<u>2</u>	<u>3</u>	<u>8</u>	<u>0</u>	<u>5</u>	<u>0</u>
<u>6</u>	<u>3</u>	<u>7</u>	<u>4</u>	<u>9</u>	<u>1</u>	<u>4</u>	<u>2</u>	<u>5</u>	<u>7</u>	<u>1</u>	<u>5</u>	<u>3</u>	<u>8</u>	<u>4</u>	<u>2</u>
<u>1</u>	<u>8</u>	<u>4</u>	<u>0</u>	<u>3</u>	<u>8</u>	<u>2</u>	<u>7</u>	<u>0</u>	<u>3</u>	<u>9</u>	<u>4</u>	<u>7</u>	<u>1</u>	<u>7</u>	<u>9</u>
				<u>9</u>	<u>3</u>	<u>8</u>	<u>5</u>	<u>4</u>	<u>9</u>	<u>3</u>	<u>6</u>	<u>9</u>	<u>7</u>		
				<u>9</u>	<u>5</u>	<u>3</u>	<u>2</u>	<u>6</u>	<u>1</u>	<u>6</u>	<u>8</u>	<u>4</u>	<u>1</u>		

2. Take a sheet of paper the width of this page and fold down one edge about $\frac{1}{2}$ inch; fold down again and again until the sheet has been used up. Then without turning the paper in any way place it under the top line of combinations in Exercise 1 and write the results; unroll one fold, slide the paper down, and write the result of the second line, and so on until all are written. Record your time in seconds for writing the results.

3. Practice counting by 7's as follows: Write in a horizontal line the numbers from 1 to 7. Note the time, using minute and second hands, and at once start counting by 7's beginning with 1, 8, 15, etc., until

you reach the 90's; under 1 write the final number in the count. Continue in the same manner, starting with each successive number. Note the time of finishing. Check by subtracting the upper number from the lower and dividing the result by 7; if each such result is exactly divisible by 7, the counting was probably correct.

4. Take Exercise 3, using every number from 2 to 9 inclusive and count each series, recording the time.

5. Add the following, using the common check, and record the time.

(a)	(b)	(c)	(d)
37	8967	906	372 896
44	9896	849	329
32	7834	94	1 428
27	4689	40	53 899
36	9784	825	97
18	8645	748	398
25	7321	54	4 864
42	9876	202	37 399
54	5383	746	892
17	6967	45	7 896
23	4682	68	426 875
<u>16</u>	<u>3847</u>	<u>822</u>	<u>327</u>

6. If time permits, it will be interesting at this point to work out in class the correlations between the abilities measured in the foregoing exercises — especially those between 1 and 3, between 1 and 5, and between 3 and 5.

7. Verify the result of 5 (b) by using the method of *partial sums*.

8. Work 5 (c) by the *civil service method* and compare results.

9. Check 5 (d) by casting out 9's.

10. Add 5 (a) by calling off results as in mental addition, thus: 37, 77, 81, etc.

11. List the combinations and decades used in finding the sum of the units' column in 5 (d).

12. The following, taken from a recent World Almanac shows the United States exports by years. Find the totals and check by adding both horizontally and vertically for the grand total.

FISCAL YEAR	IN AMERICAN VESSELS	IN FOREIGN VESSELS	TOTALS
1920	3 183 663 922	3 866 708 250	
1921	2 203 296 091	3 398 879 596	
1922	1 163 596 800	2 035 217 849	
1923	1 264 677 892	2 036 686 438	
Totals			

13. In the square below add the numbers in each horizontal line, then add the numbers in each column, then add the two diagonals. What do you observe?

63	88	74	13	8	24	53	48	34
11	9	25	51	49	35	61	89	75
52	47	36	62	87	76	12	7	26
68	84	73	18	4	23	58	44	33
19	5	21	59	45	31	69	85	71
57	46	32	67	86	72	17	6	22
64	83	78	14	3	28	54	43	38
15	1	29	55	41	39	65	81	79
56	42	37	66	82	77	16	2	27

14. Such a square as the above is called a *magic square* — the numbers used are the first 81 numbers. Can you make a magic square of the first 9 numbers? Of the first 25 numbers?

15. Verify the horizontal additions and perform the vertical additions, using the double check, in the following table.

EXPENDED IN RED CROSS SERVICE

JULY 1ST, 1923, TO JUNE 30, 1924

DOMESTIC OPERATIONS	NATIONAL ORGANIZATION	CHAPTERS	TOTAL
Service to Disabled Veterans . . .	\$1,708,000	\$2,000,000	\$3,708,000
Service to Men of the Regular Army and Navy	308,000	377,000	685,000
Disaster Relief	306,000	287,000	593,000
Enrolled Nurses' Reserve	44,000		44,000
Public Health Nursing	265,000	662,000	927,000
Home Hygiene and Care of the Sick	69,000	73,000	142,000
Nutrition	60,000	73,000	133,000
First Aid and Life Saving	216,000	72,000	288,000
Junior Red Cross	234,000	277,000	511,000
All Other Chapter Activities Including Home Service to Civilians		1,048 000	1,048,000
Other Domestic Operations — National	280,000		280,000

16. State the rule for finding the excess of 9's of a number and show how to discover it inductively using the numbers 8472, 985, and 681.

17. Add mentally 56 and 89, and state the steps. What decade is involved?

METHODS OF TEACHING

The fundamental combinations.—Many persons teach the addition combinations in the order of the size of the sums ; others take them up by tables — the ones, the twos, etc. It is not definitely known that it makes any difference which order is followed. Before taking up formal work on the combinations it is well to familiarize

1	2	1
1,	1	2,
2	3	3
1	2	3
3	2	1,
4	4	4
	etc.	

the children with the simple problem situations which give rise to addition, having them solve the problems by counting. This work is for the purpose of defining the process through experience of its utility. In teaching the combinations in addition (see p. 39) it is necessary at first to make use of a great variety of objective material, but later on it is possible — and a saving of time — to draw the conclusion that a certain combination is true after a single experiment in counting objects, or even by counting without objects. A safe guide as to the quality and quantity of objective experience necessary will be found by putting the new combination as a question and letting the children arrive at the result in their own way. Free discussion of the methods used in finding the result will serve to fix the fact in mind, and will develop a lively interest. The best primary teaching the writers have ever seen was by a teacher who consistently followed this plan. Children are much more interested in exercising their own ingenuity than they are in using, under rigid direction, the most attractive objective material.

Care should be taken not to do so much counting that the children will form the habit of depending upon counting rather than upon memory. The one purpose of the counting is to give *meaning* to the combination, and as soon as that end is reached the counting should give place to various forms of motivated drill until the correct responses come automatically.

M. A. Bailey suggests the following domino chart as a convenient device for counting material.¹ This affords a means of determining at a glance the numbers to be

¹ BAILEY, M. A. — *A Handy Book in Teaching Arithmetic*, p. 66.

added. Thus, glancing at the fifth domino (Fig. 4), the child sees the picture of five and four and by counting the dots both ways 4, 5, 6, 7, 8, 9 or 5, 6, 7, 8, 9, ascertains that 4 and 5 are 9 and that 5 and 4 are 9. If the combinations are taught in sequence, the known combinations may be expressed in numbers and the unknown in domino form.

Any tendency to count on the fingers or to make short strokes as the addition is performed should be broken up at once.

Arguments may be advanced for and against learning the combinations in tables repeated by rote. Lennes says: "Experimental evidence and common sense lead to the conclusion that the fundamental addition and multiplication combinations should be mastered separately and not in serial tables. In

practice each combination arises as a part of a more complicated process and entirely dissociated from a table."¹ While no one could take issue with his latter statement, it is not easy to understand why it necessarily follows from the former statement. If a pupil, after finding the results of new combinations by counting, is plunged at once into miscellaneous drills, he has no means at his

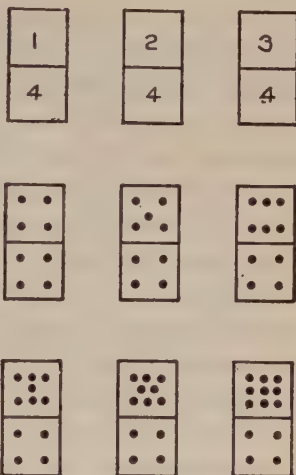


FIGURE 4

¹ LENNES, N. J. — *The Teaching of Arithmetic*, p. 213; The Macmillan Company.

command, other than counting again, for getting the results of a combination missed. It is evident that this may easily result in continuing the counting habit when performing column addition.¹

When the combinations are taught by decomposition ($16 = 8 + 8$ or $9 + 7$), the child has no means at his command for recalling the answer if he forgets the result in miscellaneous drills. If tables are learned and the child who misses a combination is asked to repeat his table, or a portion of it, until he reaches the combination needed, he sets up helpful associations for getting the result without counting. Frequently the preceding combination alone may be all that is needed. Another helpful feature of the table form is that when the combinations are learned in sequence, the number of new combinations to be learned in each table is constantly decreasing and the major part of the drill should therefore be devoted to the new combinations.

By far the greatest amount of drill must of necessity be given to calling results when the combinations are arranged in miscellaneous order. In such drills special attention should be given to the relative difficulty of the combinations. The results of investigations made along this line indicate that there is considerable variation in the number of errors made for different combinations.² Intensive drills upon those combinations that give the greatest trouble should be provided. Such combinations may be discovered by use of appropriate tests.

¹ Entering tests in arithmetic are given to high school graduates at the New York Training School for Teachers. In these tests many students resort to counting in column addition examples.

² LENNES, N. J. — *The Teaching of Arithmetic*, p. 197.

C. L. Phelps¹ lists the following difficult combinations as determined by tests given to 270 eighth-grade children :

9 + 7	9 + 5	6 + 5	7 + 4	7 + 2
8 + 5	9 + 8	7 + 5	8 + 3	5 + 3
9 + 6	8 + 4	7 + 3	7 + 6	6 + 3
9 + 3	8 + 7	8 + 8	3 + 3	9 + 2

It is noteworthy that the most difficult combinations are those involving large addends ; it is difficult, however, to account for the position of $3 + 3$. The reader should observe that the word *difficult* as here used means *causing many errors*. It is probable that with different training the order of difficulty would be different.

The decades. — It is a good method to follow each combination with its so-called *decade* combinations — 5 and 2 with 15 and 2, 25 and 2, etc. The fact that the children soon learn to run through the decades without assistance as soon as they know the fundamental combinations is strong evidence as to the soundness of the procedure.

The decade combinations are first taken up in serial order (following $\frac{4}{7}$ with $\frac{4}{17}, \frac{4}{27}$, etc.), but soon drill work on them in miscellaneous order should be given. Since these combinations find their principal application in connection with column addition, and since in this connection the two-order addend always comes first and is held in mind rather than presented to the eye, it is good practice to dispense with the combinations in their written form as soon as practicable, and give purely oral drills on the same.

¹ PHELPS, C. L. — "A Study of Errors in Tests in Adding Ability," *Elementary School Teacher*; September, 1913.

This is in harmony with the general principle that practice upon a given habit should be carried on under conditions as nearly as possible like those in which the habit is to function.

A frequent error in decade additions in a column addition example is to call off for the tens of the answer two more tens than in the number increased (see page 29). Hence a necessity arises for stressing the point in decade addition that the tens of the answer will be either *the same or one more ten*. The following device is helpful:

Use a chart containing all the numbers of two orders arranged in sequence.

10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

As numbers are pointed to in miscellaneous order and increased by a constant addend, such as 6, the pupil may be asked to tell whether the tens will be the same or one more. Thus when 27 is indicated, the answer is "thirty-something"; when 41 is pointed to, the answer is "forty-something"; etc. The units of the answer, obtained by association with the corresponding combination, are then combined with the tens and the entire answer is called.

The forty-four counting series (see page 30) need not, and should not, be taken up until the children have gained a fair control of the decades organized according to *endings* ($\begin{smallmatrix} 4 & 4 \\ 7 & 17 \end{smallmatrix}$, etc.). As the children advance they

become more capable of the sustained attention required in such counting. Do not require the children to take too many steps at a time. At first counting by 3's to 20, or from 20 to 41, or through any similar range is enough. Such work should *move*; it is a mistake to allow a child to flounder hopelessly, to count on his fingers or otherwise. It is better that he should defer this form of practice until he is surer of himself.

All such work has a rhythmic character. Some teachers employ rhythm consciously for the purpose of securing heightened effort. Unfortunately, not all people can set an even rhythm; when a teacher beats time poorly, it is a painful distraction to the child. The authors suggest the use of a pendulum — a bright cord with an attractive weight — so hung that the length of the cord may be changed at will. This device has the advantages of silence, impersonality, and regularity of tempo. It would be better for the children to adjust the length of the pendulum to suit themselves than to feel the nervous strain of the teacher's pace setting. The utility of such a device can be accurately measured by the pupil's demand for it.

Speed in this work can be stimulated by having the child compare his speed in counting by 3's with his speed in counting by 1's; also by having the pupils see who can write the most results in counting by 3's in one minute.

Column addition. — *Carrying* in addition is usually explained rationally, though some teachers present the process in a purely mechanical fashion. If the children have used the cups and sticks in their study of two-order numbers (see page 17), it will be easy to explain *carrying* by referring to this objective experience. A mere verbal

explanation, however exact mathematically and however skillfully done, is quite sure to miscarry. If the child is to learn anything about the nature of the process, it must be through concrete experience of some sort in which he is the actor. Before western peoples learned the Hindu notation all their work in computation beyond the simple

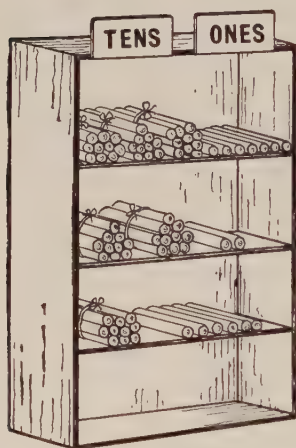


FIGURE 5

combinations was carried on with the abacus — that is, with objects. It is not at all certain that the apparently rapid progress resulting from the teaching of the mere mechanics of a process is not more than offset by the attendant neglect of opportunities to give children a grasp of number relations. The chief aim in the study of arithmetic is not mere facility with *figures* but an easy control of *numbers* — a sense for quantity and relations.

The writers have seen the device in Fig. 5 used to good advantage in giving children insight into the meaning of column addition; an egg carton (3×4) might be used for the purpose. The example shown would be objectified by placing in compartments under *ones*, 7 sticks, 2 sticks, and 6 sticks; and under *tens*, 3 bundles, 2 bundles, and 1 bundle. Uniting the sticks under *ones*, we should get 15, which make a bundle of ten and leave 5 sticks in the sum under *ones*; combining the tens would give 7 tens in the sum under *tens*.

37
22
16
—

Each step in the objective work must be worked out with the abstract symbols.

This device should, of course, be used only under conditions in which the teacher is convinced that it will shorten the road to success. Probably it would be of no value in dealing with the more capable children.

Frequent practice on the *working explanation* (p. 31) is extremely valuable for the purpose of controlling the children's procedure until the desired habits have been firmly established. Many children fail to coördinate the movements of eye and hand with the mental steps in making the combinations, or they add something when a 0 is encountered, or they fail to keep to the column when the addends have varying numbers of places. These, and many other, special difficulties arise with certain individuals. The teacher must see to it that they occur in examples to be worked orally before they occur in silent written work. These examples are not to be regarded as distinct types to be formally taught; instruction is required only when the children get into difficulties, and then should be made as pointed as possible.

A rather common error is to write the left-hand figure of a column total and carry the right-hand figure. If the children are required from the outset to write the column totals at the right, as in the common proof (p. 32), this error will be readily overcome, if not avoided entirely; because writing and seeing the number helps them to distinguish the parts more clearly — hence to handle them with greater assurance.

Checking results. — Many teachers have expressed the view that young children are unable to understand the

necessity for accuracy, and that over-insistence upon accuracy is therefore a mistake. In reply it may be said that young children are able to understand completely very few of the things they are required to do in school. They have by nature great confidence in their teachers, and if their school activities are on the whole successful, they will enjoy these activities without serious questions as to their utility. Why should a child be allowed to think of an example with a *wrong answer* as having been *worked at all*? On this point Thorndike,¹ under the head of "Early Mastery," says: "The bonds in question must be made far stronger than they now are. . . . Any bond should be made to work perfectly, though slowly, very soon after its formation is begun. Speed can be easily added by proper practice." And he mentions as one of the chief reasons for inaccuracy — "The pupil is not taught to check his work."

Of the various forms of the common check (p. 32) the most useful is the one in which the column totals are written at the right of the example. The other forms, and also the check by 9's, may be introduced in the 7th or 8th year of school.

EXERCISES

1. What are the 45 combinations in addition? The 55? The 100?
2. State the steps in learning the combinations and illustrate how each may be taken.
3. What devices should be used in breaking up the tendency to count in giving the results for the combinations?
4. List all the combinations and decades in the following example. Give the full or teacher's explanation; the working or pupil's explanation.

¹ THORNDIKE, E. L. — *The Psychology of Arithmetic*, p. 105; The Macmillan Company.

97

68

94

37

5. State the two steps in learning the *decades* and show how to take each step.

6. Write the addition combinations, arranging them according to their sums — all whose sums are 2, all whose sums are 3, etc. Can you think of any advantages of teaching them in this order? Any disadvantages?

7. Some teachers teach the zero combinations along with the others; other teachers defer them until they arise in column addition. Discuss the relative merits of these plans. In and of itself, would the combination $5 + 0$ have any meaning to a child?

8. Assuming that the table $\begin{array}{cccccc} 1 & 1 & 2 & & 1 & 9 \\ \hline 1 & 2 & 1 & \dots & 9 & 1 \\ \hline \end{array}$ has been learned, work out the first lesson on the table of twos.

9. Assuming that you are teaching the combinations in the order of their sums, work out the first lesson on the combinations whose sums are 5.

10. In what respects would a lesson on new addition combinations in the second year differ from a corresponding lesson in the first year? Outline a procedure for such a second-year lesson. Take the table of 6's.

11. Tell briefly how you would proceed in teaching the decade combinations which follow the combination $\frac{6}{7}$.

12. Conduct a drill on the decades connected with 5 and 4; with 3 and 8. Show which set is the more difficult for beginners.

13. (a) What decade drill is needed for a learner who fails on $36 + 7$? (b) In what counting series does this decade occur?

14. Work out the details of teaching carrying from units to tens. Would you teach carrying from tens to hundreds in the same way? Explain your answer.

15. In teaching mental addition of numbers of two orders, which would you present first, $46 + 37$ or $46 + 30$? How would you proceed in the early work in establishing the correct habit?

CHAPTER III

SUBTRACTION

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions. — We have considered the need of finding the number of things in a whole from the number of things in the groups that may be combined to make the whole. Let us now consider a different possibility. The number of things in the whole and the number of things in one of its component groups may be known and we may desire to find the number of things in the other group. Three types of problems occur which give rise to this new process, *subtraction*.

1. John had 10¢ and spent 8¢; what had he left? Here the number in both groups is the number of cents that he had at first, the number in the known group is the number that he spent, and the number in the unknown group is the number left.

2. John has 8¢ and he wishes to buy a ball for 10¢; how many more cents does he need? Here the number in the whole is the number of cents that the ball costs, the number in the known group is the number of cents he has, and the number in the unknown group is the number that he needs.

3. John has 10 marbles and Henry has 8 marbles; how many more has John than Henry? Here, the number in the whole is the number that John has, the number in the known group is the number that Henry has, and the number in the unknown group is the number by which John's marbles exceed Henry's.

Type 1 involves finding what is left when one number is taken from another; type 2 what must be added to one number to produce another, and type 3 how much greater one number is than another. The definition of subtraction must be general enough to include these three aspects.

Subtraction is the process of finding the number of things in one of two groups from the number of things in both groups and the number of things in the other group.

That subtraction is the inverse of addition is seen from the following: In the equation $6¢ + 4¢ = 10¢$, the $6¢$ may be wanting, or the $4¢$ may be wanting; that is, if one of the two addends and their sum are given, the other addend may be found.

$$6¢ + 4¢ = 10¢$$

$$? + 4¢ = 10¢$$

$$6¢ + ? = 10¢$$

Hence, subtraction may be defined as the process of finding one of two addends from their sum and the other addend. The number which corresponds to the sum, *i.e.*, the number from which we subtract another is called the *minuend*, from *minuere*, to lessen, *minuendum*, to be lessened; the number to be subtracted is called the *subtrahend* and the result obtained by subtracting is called the *difference* or *remainder*.

The difference between two numbers may be expressed by the sign ($-$), which is read *minus*. For example: $5 - 2$ means that 2 is to be subtracted from 5; it is read: *five minus two*.

Elements in mastery of subtraction. — To master the process of subtraction, one should understand the fundamental methods of performing it, the types of examples

in order of progressive difficulty, and the best procedure for solving each type.

You may be interested in trying the following standard test ¹ to ascertain your standing in subtraction. The time allowed is 1 minute :

616	1248	1365	1092	716
<u>456</u>	<u>709</u>	<u>618</u>	<u>472</u>	<u>344</u>
1267	1335	707	816	1157
<u>509</u>	<u>419</u>	<u>277</u>	<u>335</u>	<u>908</u>
1355	908	519	1236	1344
<u>616</u>	<u>258</u>	<u>324</u>	<u>908</u>	<u>818</u>
1009	768	1269	615	854
<u>269</u>	<u>295</u>	<u>772</u>	<u>527</u>	<u>286</u>

The median score for St. Louis pupils of the 8th grade is 11.3. How much better can you do?

Subtraction by counting. — The fundamental method of performing subtraction is by counting objects. Using the problems of a previous section, we have :

1. John had one, two, . . . ten cents ; he spent one, two, . . . eight cents (crossing out one cent at each count) : he has one, two cents left.

∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅

2. John has one, two, . . . eight cents ; he needs nine, ten cents (placing in a separate group a cent at each count) : he needs one, two cents more.

○ ○ ○ ○ ○ ○ ○ ○ ○ ○

3. John has one, two, . . . ten marbles ; Henry has one, two . . . eight marbles ; John has one, two more than Henry.

○ ○ ○ ○ ○ ○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○ ○ ○

¹ Cleveland Survey Test. — Set F.

The combinations. — A more advanced way of subtracting is to find the result through a knowledge of addition. Thus, in subtracting 9 from 12, we find the result, 3, by considering what number added to 9 will give 12.

The simplest type of example in subtraction is that which arises from the 45 combinations in addition. There are 90 such combinations in which numbers of one order, when subtracted from numbers ending in each of the ten digits, give remainders of one order :

1 2 3 . . . 10	2 3 4 . . . 11	3 4 5 . . . 12
<u>1 1 1</u> <u>1</u>	<u>2 2 2</u> <u>2</u>	<u>3 3 3</u> <u>3</u>
4 5 6 . . . 13	5 6 7 . . . 14	6 7 8 . . . 15
<u>4 4 4</u> <u>4</u>	<u>5 5 5</u> <u>5</u>	<u>6 6 6</u> <u>6</u>
7 8 9 . . . 16	8 9 10 . . . 17	9 10 11 . . . 18
<u>7 7 7</u> <u>7</u>	<u>8 8 8</u> <u>8</u>	<u>0 0 9</u> <u>9</u>
	0 1 2 9	
	<u>0 0 0 . . . 0</u>	

Including the zeros there are ten more.

To master the 100 combinations in subtraction so that the results can be given instantly when the combination is presented is the foundation work in subtraction. There are two steps necessary :

(1) Find the result by referring to the corresponding addition combination.

(2) Fix the result in mind.

In purely abstract examples it is necessary to choose which idea of subtraction shall be employed. The "take away" idea and the "add to" idea are both in

common use. For example, the answer to the combination $\begin{array}{r} 15 \\ - 8 \end{array}$ may be declared by saying 8 from 15, 7, or by saying $\overline{8}$ and 7 are 15. In fixing the results in mind it is not necessary to memorize them in sequence, that is, in table form, because if a combination is missed, it may be recalled by referring to the corresponding addition combination.

Column subtraction. — The easiest type of example is that in which each figure of the minuend is equal to or greater than the figure in the same order of the subtrahend. In such examples we subtract units from units, tens from tens, etc., declaring the combinations in either manner described above. In the example $\begin{array}{r} 86 \\ 24 \end{array}$, say 4 and 2 are 6; 2 and 6 are 8; or, 4 from 6, 2; 2 from 8, 6.

The more difficult type of example is that in which at least one figure in the minuend is less than the figure in the same order of the subtrahend. For example, subtract 378 from 542.

Since it is impossible to subtract 8 units from 2 units or 7 tens from 4 tens, some method must be employed to make the subtraction possible. Three different methods have been in use for centuries — two of them commonly known as the Italian methods, and the third as the Austrian method. Each depends upon a different principle. A teacher should understand the principle and be able to give a concrete explanation in applying it; a pupil should not be hampered in his work by a long explanation.¹

¹ See footnote on page 31.

First Italian method. — *Principle.* — One may be taken from any order of a number and its equivalent added to the next lower order without affecting the value of the number. Thus, in 42, we may take 1 ten from 4 tens and add its equivalent, 10 units, to 2 units, making 12 units. 3 tens and 12 units is equal in value to 42.

Complete or teacher's explanation:

542	4	13	12
<u>378</u>	<u>3</u>	<u>7</u>	<u>8</u>

Since 8 units cannot be taken from 2 units, 1 ten is taken from 4 tens, leaving 3 tens, and its equivalent, 10 units, is added to 2 units, giving 12 units; 8 units from 12 units, 4 units. 7 tens cannot be taken from 3 tens; 1 hundred is taken from 5 hundreds, leaving 4 hundreds, and its equivalent 10 tens is added to 3 tens, giving 13 tens; 7 tens from 13 tens, 6 tens; 3 hundreds from 4 hundreds, 1 hundred. Hence the remainder is 164.

Working explanation. — 8 from 12, 4; 7 from 13, 6; 3 from 4, 1.

Second Italian method. — *Principle.* — Adding the same number to both minuend and subtrahend does not affect the remainder. Since 10 of one order is equal to one of the next higher, we add 10 to any order of the minuend and one to the next higher order of the subtrahend.

Complete or teacher's explanation:

542	5	14	12
<u>378</u>	<u>4</u>	<u>8</u>	<u>8</u>

Since 8 units cannot be taken from 2 units, 10 units are added to 2 units, giving 12 units, and 1 ten to 7 tens,

giving 8 tens ; 8 tens cannot be taken from 4 tens, 10 tens are added to 4 tens, giving 14 tens, and 1 hundred to 3 hundreds, giving 4 hundreds ; 8 units from 12 units, 4 units ; 8 tens from 14 tens, 6 tens ; 4 hundreds from 5 hundreds, 1 hundred. Hence the remainder is 164.

Working explanation. — 8 from 12, 4 ; 8 from 14, 6 ; 4 from 5, 1.

Austrian method. — *Principle.* — The sum of the subtrahend and the remainder equals the minuend.

Complete explanation. — What must be added to 378 to equal 542?

$$\begin{array}{r} 542 \\ 378 \\ \hline \end{array}$$

$$\begin{array}{r} 542 \\ 378 \\ \hline 4 \end{array}$$

$$\begin{array}{r} 542 \\ 382 \\ \hline 60 \end{array}$$

$$\begin{array}{r} 542 \\ 442 \\ \hline 100 \end{array}$$

$$\begin{array}{r} 542 \\ 378 \\ \hline 164 \end{array}$$

To 8 units a number of one order must be added to give a number ending in 2, or 12. 8 units and 4 units are 12 units. We write the 4 in units' place. Adding 4 to 378 gives 382. We must now add a number of one order to the 8 in tens' place to give a number ending in 4 in tens' place, or 14. 8 tens and 6 tens are 14 tens ; we write the 6 in tens' place. Adding the 6 tens or 60 to 382 gives 442. We must now add enough to 4 hundreds to give 5 hundreds. We write the 1 in hundreds' place. We have now added 4, 60, and 100 or 164 to 378 to give 542.

Working explanation. — 8 and 4 are 12 ; 8 and 6 are 14 ; 4 and 1 are 5.

Special applications of Austrian method. — The Austrian method has some interesting and useful applications.

It may be used to find a missing addend where the sum and several addends are given. Illustration : Find the

difference between 4120 and the sum of 743, 826, 497, 868, 349.

743 The working explanation is as follows,
 826 adding from the top down: 9, 16, 24, 33,
 497 40√ and 7 are 40; write 7, carry 4; 8, 10, 19,
 868 32√ 25, 29, and 3 are 32; write 3, carry 3; 10,
 349 41√ 18, 22, 30, 33, and 8 are 41. Write 8.
 (837) Prove by adding up.

4120

In balancing an account the balance may be found by finding what number added to the numbers on the lesser side will equal the sum of the numbers on the greater side.

Illustration: Balance the following account:

(a)		(b)	
Dr.	Cr.	Dr.	Cr.
86.37	68.72	86.37	68.72
52.95	59.87	52.95	59.87
186.84	13.29	186.84	13.29
	28.36		28.36
			Bal. 155.92
		<u>326.16</u>	<u>326.16</u>
		Brought for'd	155.92

We must find the sum of the greater side, which in this example is the Dr., and then determine what number added to the Cr. side will equal this amount. (That is, find the missing addend on the credit side.) Hence, the procedure is the same as for finding any missing addend.

The work is arranged as shown above in (b).

The working explanation for finding the balance is: 9, 18, 24, and 2 are 26 (write 2, carry 2); 9, 17, 19, 22, and 9 are 31 (write 9, carry 3); etc.

In a long-division example the multiplication and subtraction may be performed together ; this is known as the Austrian method of division (see page 115).

Proofs or checks. — *The common proof* is to add the subtrahend and the remainder ; the result should equal the minuend. If the Austrian method is used, the reverse of the combinations made in finding the result should be used in the proof.

In finding the result one says 8 and 4 are 12 ; in proving

$$\begin{array}{r} 542\checkmark \\ 268 \\ \hline 274 \end{array}$$
 he says, 4 and 8 are 12 ; 8 and 6 are 14 ; 3 and 2 are 5. Check the minuend if the answer is correct.

Proof by nines. — Add the excesses of 9's of the subtrahend and remainder ; the excess of 9's in this result should equal the excess of 9's of the minuend. Since the sum of the subtrahend and remainder equals the minuend, the proof of the rule is similar to the proof of the rule for addition. •Illustration :

$$\begin{array}{r} 528 \quad 6\checkmark \\ 259 \quad 7 \\ \hline 269 \quad 8 \end{array}$$

Check : 8 and 7 are 15 ; the excess of 9's in 15 is 6.

Mental subtraction. — To subtract a number of two orders from a number of two or three orders, mentally, subtract the tens of the subtrahend from the entire minuend, and from this result subtract the units of the subtrahend.

Illustration : $94 - 38$; $94 - 30 = 64$; $64 - 8 = 56$.

Working explanation. — The pupil calls off the minuend and the successive results ; no other words should be spoken, thus : 94, 64, 56.

HISTORICAL NOTES

Early methods. — In early times subtraction, like addition, was performed by means of the abacus. The Austrian method was so named by the Germans, who obtained it from the Austrians. They seem to have been the first to adopt this method,¹ although it was probably not originated by them. The two Italian methods were in early use among the Hindus,² although Cajori says the name "Italian method" has been traced back to an English arithmetic of 1730.³

Ball says that the earliest use of the sign $(-)$ of which we have any knowledge occurs in the fifteenth century in a mercantile arithmetic, written by Johannes Widman.⁴ Various explanations are given as to the probable origin of the symbol.⁵

EXERCISES

1. What is the derivation of the word *subtrahend*? Subtract?
2. Explain in detail the difference between the solutions by counting of the problems on page 52. Give three original problems illustrating these types and solve by counting.
3. Write the 100 combinations in subtraction in miscellaneous order as in Exercise 1 (p. 36). Call off results orally; then write results on folded paper according to directions given in Exercise 2 (p. 36). Keep a record of your time for each.
4. For each of the following examples give: (a) the working explanation; (b) the full explanation by each of the three methods mentioned in this chapter:

376	362	743	743	8000
<u>124</u>	<u>148</u>	<u>266</u>	<u>368</u>	<u>1468</u>

¹ CAJORI, F. — *A History of Elementary Mathematics*, p. 213; The Macmillan Company.

² CAJORI. — *Op. cit.*, p. 96.

³ CAJORI. — *Op. cit.*, p. 213.

⁴ BALL, W. W. R. — *A Short History of Mathematics*, p. 212; The Macmillan Company.

⁵ BALL. — *Op. cit.*, p. 213.

5. Subtract and give the working explanation by each of the three different methods, proving each answer :

5289683

5982175504

427894321

92349876

2479847299708439525399674882938057

6. Give the working explanation for the common proof of the examples in question 4.

7. Subtract mentally, using the working explanation: 65 from 111; 84 from 123; 17 from 71; 47 from 96; 15 from 92.

8. The sum of the discount, the cost of collection and exchange, and the proceeds equals the face of a note. Complete the following bank record of notes discounted (see pages 473-476) :

FACE	DISCOUNT	COLL. AND EXCH.	PROCEEDS
927.18	5.24	.65	
528.22	4.37	.34	
568.91	4.88	.41	
568.91	4.88	.59	
750.00	7.48	.74	

9. Find for each year the value added by manufacturing :

YEAR	COST OF MATERIAL	VALUE OF PRODUCT	VALUE ADDED
1919	29 775 816	49 962 388	
1914	11 533 547	19 554 773	
1909	9 912 122	16 827 427	
1904	6 985 816	12 134 078	
Totals			

10. Explain how to balance the following account, using the Austrian method :

SUBTRACTION

61

J. B. SWAN AND COMPANY

1924							
Jan.	1	To Mdse.	478.28	Jan.	10	By Check	385.75
	7	To Mdse.	165.47		18	By Check	168.
	16	To Mdse.	357.		23	Mdse. ret.	326.95
	24	To Mdse.	275.85			Balance	
Feb.	1	Bal. for'd					

11. This table shows for a recent year the migrations of European people to and from the port of New York.

ORIGIN	IMMIGRANTS	EMIGRANTS	NET GAIN
Bohemia, Moravia	6,380	811	
Bulgaria, Serbia, Montenegro	1,841	1,162	
Croatia and Slovenia	2,961	42	
Dalmatia, Bosnia, and Herzegovina	225	116	
Holland	6,185	553	
England	58,754	4,041	
Finland	3,351	208	
France	24,663	770	
Germany	82,062	792	
Greece	4,380	4,015	
Ireland	31,468	920	
Italy	44,196	13,087	
Lithuania	1,808	255	
Poland	17,888	1,585	
Portugal	3,127	2,555	
Rumania	1,323	645	
Russia	8,056	435	
Scandinavian Countries	32,445	1,369	
Scotland	46,626	728	
Spain	2,337	2,068	
East Indies	85	102	

62 AN ARITHMETIC FOR TEACHERS

- (a) Find the net gains by countries.
- (b) Find the total of each column.
- (c) Check by subtracting total emigrants from total immigrants and comparing with total net gain.

NOTE. — One item under *net gain* is a minus quantity. Indicate by sign (—).

12. (a) Find the missing addends, then prove by adding in opposite direction.

3874	348	786539
5968	756	837453
7532	59	835621
()	237	264732
<u>20861</u>	835	27839
	536	893208
	637	28391
	734	123456
	78	5674
	786	34861
	()	()
	<u>9876</u>	<u>6382018</u>

- (b) Give working explanation for each of the above.

13. Balance the following accounts, arranging in proper form :

DR.	CR.	DR.	CR.
761.39	324.96	6.34	100.00
27.68	82.57	19.25	
14.95	56.91	7.88	

14. Give the working explanations for Exercise 13.
15. Discover inductively (a) the principle of the first Italian method, (b) the principle of the second Italian method.
16. Discover inductively the rule for proving subtraction by 9's.

METHODS OF TEACHING

The combinations. — The order of the subtraction combinations should correspond with the order in addition, because the former are derived from the latter. Here again we should start with interesting problem situations for purposes of definition. If the Austrian, or additive, method of subtraction is to be used, these problems should be of the *additive* type; *e.g.*, Henry wishes to buy a top for five cents. He has only three cents. He must ask his mother for — more cents. If any other method of subtraction is to be used, the problems should, of course, be of the *take-away* type. It will be helpful, for a time, to get the results by counting, but as soon as practicable this method should give place to the direct use of the addition combinations. If the children are accustomed to the circle device in drill on the addition combinations the following procedure is effective.

1. Give drill on the addition combinations until they are clearly in mind, finally writing the sums outside the circle (Fig. 6). The children say: *Three and six are nine*, etc.



FIGURE 6

2. The device now looks like Fig. 7. Ask the children to cover their eyes; erase one of the figures within the circle and ask: *What have I erased?* Some child answers: *four*. The teacher asks: *How do you know?* Answer: *Because 3 and 4 are 7*. Proceed in like manner with the other combinations. As suggested in addition, special attention must be given to the more difficult combinations.

3. The device now looks like Fig. 8. Now take up drill on the combinations, the children saying, *3 and 9 are 12*, *3 and 1 are 4*, etc.

4. Next write the combinations in column form, thus:

12	4	7	10	6	11	5	8	3	9
<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>

Have the children practice as in step 3.

The above steps are given in great detail because the unfavorable results sometimes obtained with the Austrian method are due to poor technique in teaching. Very few teachers were brought up on this method, and to many it seems awkward and unnatural.

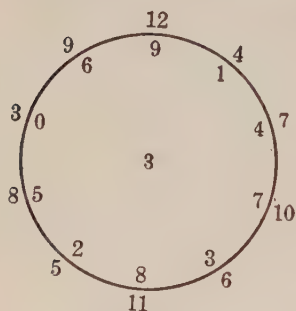


FIGURE 7

The customary procedure in teaching the combinations in their *take-away* sense is essentially the same as that outlined above until step 3 is reached; then it is necessary to introduce

the *take-away* idea — a rather difficult matter, since it reverses the thought of addition. Some teachers get all the subtraction facts by means of counting. This is a possible procedure but it is wasteful of time and effort, failing, as it does, to take advantage of the close and obvious relationship between addition and subtraction. As is usual with words and symbols, the meaning which the children attach to the word *minus*, and the sign ($-$), will be in harmony with their mental experience in

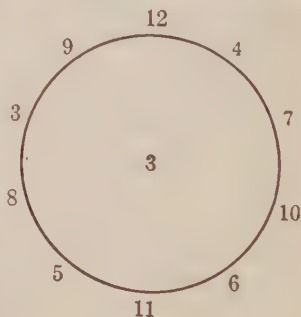


FIGURE 8

the process of subtraction. The fact of the Latin derivation of *minus* does not bother them at all. Where the Austrian method is in use it is well to defer the word *minus* until the combinations have been mastered; the sign need not be used until there is occasion to express subtraction in the horizontal form, $13 - 6$.

Written or column subtraction. — On page 54 the two general types of examples are given. But for teaching purposes it may be helpful to separate each of these into sub-types as follows:

TYPE I

(a)	(b)	(c)	(d)	(e)	(f)
96	136	136	136	130	375
<u>64</u>	<u>84</u>	<u>86</u>	<u>80</u>	<u>80</u>	<u>42</u>

In I (a) we have two simple combinations. In I (b) there are more figures in the minuend. In I (c) we have 0 in the units' order of the remainder. In I (d) 0 occurs in units' order of the subtrahend. In I (e) 0 occurs in units' order of all the terms. In I (f) 0 (unwritten) occurs in hundreds' order of the subtrahend.

In teaching the sub-types under Type I the most straightforward procedure is the best. Using the working explanation, demonstrate the process and have the children imitate. Do not require, or even permit, children to do silent seat-work until you are pretty sure they know just how to proceed; disregard of this will result in *practice in error*.

TYPE II

(a)	(b)	(c)	(d)	(e)	(f)
746	746	746	740	708	700
<u>328</u>	<u>382</u>	<u>378</u>	<u>328</u>	<u>326</u>	<u>326</u>

The sub-types under II are classified according to where the borrowing, or carrying, occurs; they also present certain new zero difficulties. Detailed descriptions of these sub-types are left to the student.

The difficulty presented in Type II (a) requires very careful teaching. Though the teacher must know the complete explanation of the process thoroughly, she should not make the mistake of depending upon a mere verbal explanation in her teaching. The aim is to help the children to master the procedure. The reasoning underlying the procedure, if presented in some concrete way, may help, probably will help, *some* of the children; but it is a mistake to carry any explanation or concrete illustration beyond the point of its practical utility as an aid to the launching of the new habit. It is helpful, at times, to show the children a mixed list of examples (in their books or on the board) and have them tell in which ones they will have to carry, or change the tens. This will serve to focalize the attention upon the new difficulty.

Full explanations by the children should seldom, if ever, be required, but daily practice on the working explanation (p. 56) should be given until the correct mental habit is firmly established. Do not assign home work involving a new habit until the habit is fixed. This rule is highly important in connection with the work in subtraction, because of the several different methods in vogue. A child who learns one method in school and another method at home is almost certain to become confused between the two and hence fall short of the skill he might otherwise be expected to gain.

In the first stages of the instruction it is necessary to amplify, in some manner, the written work in order to make the nature of the process clear. The amplified forms shown under (a) below are not good, because on account of their simplicity they tend to become habitual. A considerable number of the students who enter normal and training classes practice this habit — not because they need such a crutch, but because they were either taught, or at least permitted, to use it. The forms under (b) make the processes clear, and, because of their elaborateness, are not likely to become habitual; the worse a crutch¹ is, the better.

	(a)	(b)
First	56^{11}	$61 = 5 \text{ tens} + 11 \text{ units}$
Italian	37	$37 = 3 \text{ tens} + 7 \text{ units}$
Method		$24 = 2 \text{ tens} + 4 \text{ units}$
Second	6^{11}	$61 + 10 = 6 \text{ tens} + 11 \text{ units}$
Italian	37	$37 + 10 = 4 \text{ tens} + 7 \text{ units}$
Method		$24 \qquad 2 \text{ tens} + 4 \text{ units}$

It may be helpful to some pupils to represent objectively the subtraction process, using the device (p. 46) shown in connection with teaching carrying in addition.

If the Austrian method is taught, it is not necessary to amplify the written work, but if the teacher desires to use an amplified form, the form on page 56 may be used.

The steps of the Austrian method may be shown objec-

¹ "Since it is undesirable that a pupil should regard a 'crutch' response as essential to the total procedure, or become so used to it that he will be disturbed by its absence later, it is supposed that the bond between the situation and the crutch should not be fully formed. There is a better way out of the difficulty, in case crutches are used at all. This is to associate the crutch with a special 'set' and its non-use with the general set which is to be the permanent one." THORNDIKE. — *The Psychology of Arithmetic*, p. 112.

tively, and just as vividly and satisfactorily as those of either of the Italian methods. Take the example $\begin{array}{r} 72 \\ 45 \cdot \end{array}$

This means that we must add enough to 45 to make 72.

Place 45 sticks in the cups as in (a). We shall first find what must be added to 5 to give a number ending in 2; it must be 7, as 5 and 7 are 12. We put these 7 sticks in units' cup as in (b), making 12. Of ten of these we

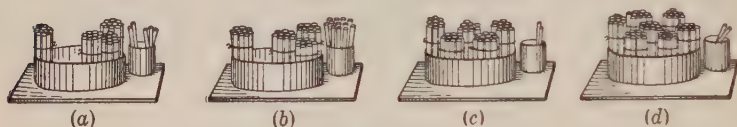


FIGURE 9

make a bundle for tens' cup, giving 52 as in (c). But our sum is to be 72, so we must add enough to 5 tens to make 7 tens. We add 2 tens as in (d). In the process we have added 7 units and 2 tens. Hence the answer is 27.

Methods compared. — The question, Which is the best method of subtraction? has been widely discussed. Unfortunately, most of the experiments have been upon children whose habits were well established, hence they ignore the time element. It is the authors' opinion, based upon experiment, that the Austrian method is superior for the reason that the thought of the children moves in the same direction as in addition. This results in a very considerable economy of time and effort. In other words, the *bonds*, to use a word much used by Thorndike, which function in addition require very slight modification to function in the Austrian subtraction as

compared with the modification necessary to function in any other method.

There are certain *a priori* reasons in favor of the Austrian method.

1. It harmonizes with the common practice in making change.

2. It simplifies (and no other method does) the process of finding one of several addends — so useful in accounting (see page 57) and other business forms.

3. The complications that arise in examples such as 8000
1267 (especially when the first Italian method is used)

are avoided entirely when 8000 is regarded as a *sum* to be obtained by adding something to 1267. This example is possibly easier than $\frac{8132}{1267}$, and certainly no harder.

4. It may be used in long division¹ (see page 115).

As between the two Italian methods there is one reason, at least, for preferring the second. In examples like $\frac{8000}{1267}$ it is manifestly easier to make the three pairs of compensating changes than it is to "decompose" 8000 into 799(10).

J. T. Johnson² tested 277 students in Chicago Normal College, using the examples on lesson card 33 of the Courtis Standard Practice Tests, Series B. All were asked to finish the examples and record their time. Of these students 220 used the first Italian method (see page 55),

¹ This method of procedure is commonly taught in French elementary schools. See, for example, Ch. Joseph's *Petit Cours d'Arithmétique, Première Partie*; Librairie Hachette, Paris.

² JOHNSON, J. T. — "The Merits of Different Methods of Subtraction," *Journal of Educational Research*; November, 1924.

23 used the second Italian method (see page 55), 18 borrowed as in the first Italian method and then handled the combinations by the additive method, 13 used the purely additive method (see page 56), and 13 used mixed or freak methods. The subjects were examined in eleven groups. The results are shown in the following table.

METHOD	MEDIAN TIME IN MINUTES	AVERAGE RIGHTS SCORE	PER CENT ACCURACY
First Italian	2:55	15.31	90
Second Italian	2:20	16.217	95.4
Borrowing, with Com- binations Additive . .	2:35	15.75	92.6
Pure Austrian	2:00	16.384	96.3
Mixed and Freak . . .	3:40	15.461	90.9

In all of the eleven groups examined, except one, the second Italian and Austrian methods made better scores in both speed and accuracy than the first Italian method.

W. H. Winch¹ (London, England) experimented with a class of eleven-year-old children who had been trained in the first Italian method. On the basis of a preliminary test he divided the class into two groups equal in number and in subtraction ability. One group was given daily instruction and practice on the second Italian method and the other group was given the same time on the old method, both groups having the same teacher. After a period of five weeks' careful testing he showed that:

(1) The method of equal additions in subtraction taught to children late in school life, who have hitherto worked by decomposition, pro-

¹ WINCH, W. H. — "Equal Additions *versus* Decomposition in Teaching Subtraction"; *Journal of Experimental Pedagogy*, 1920.

duces results in a few weeks equal on the whole, and superior in the weaker children, to those produced by the method of decomposition.

(2) The amount of gain involved does not justify a change of method at this late period of a child's school career.

Winch supervised a similar experiment with eight-year-old children who had done very little work in subtraction, some of whom could not subtract at all. The two groups were given a series of eight lessons of 20 minutes each, and then tests to measure improvement. The group using equal additions improved about twice as much as those who used decomposition; the initial and final scores were as follows: 134.9 and 230.8 for the equal additions group, as against 136.1 and 178.2 for the decomposition group.

P. B. Ballard¹ gave uniform tests in the four operations in 23 schools using the second Italian method and in 48 schools using the first Italian method. For every age group he found the second Italian method superior to the other, the difference between the two being greatest at the beginning and diminishing as the children get older; the superiority of the second Italian method varies from 40 per cent to 10 per cent. This marked superiority in subtraction is not accompanied by a corresponding superiority in the other fundamental operations. For example, the following table compares the scores of nine-year-old boys in the two groups of schools. These boys who are more than 100 per cent better in subtraction are less able on the whole in the other operations. Their measurable superiority in division may well be accounted for as the effect of superior subtraction ability.

¹ BALLARD, P. B. — "Norms of Performance in the Fundamental Processes of Arithmetic"; *Journal of Experimental Pedagogy*, December, 1924.

SCORES OF NINE-YEAR-OLD BOYS

	ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION
Using First Italian Subtraction	20.72	21.11	45.67	17.67
Using Second Italian Subtraction	20.48	45.31	38.27	19.00

The following table shows the differences between the averages in the schools using equal additions (E A schools) and the corresponding averages in the schools using the decomposition method (D schools). It shows that while

E A SCHOOLS' AVERAGE MINUS D SCHOOLS' AVERAGE

	ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION
Boys. . . .	1.41	9.73	4.08	1.50
Girls. . . .	-.02	7.42	.68	-.67

these schools differ little in ability in the other operations, they differ widely in subtraction ability.

W. W. McClelland¹ also found the method of equal additions measurably superior to the decomposition method in both speed and accuracy. His conclusions are based on a series of tests given over a period of twenty weeks to 143 children from 12½ to 13½ years of age.

A study made by one of the authors tends to show that

¹ MCCLELLAND, W. W. — "An Experimental Study of the Different Methods of Subtraction"; *Journal of Experimental Pedagogy*, December, 1918.

the Austrian method is more easily learned than are the *take-away* methods.¹

The experiments above cited indicate that the first Italian method (method of decomposition) is distinctly inferior to the other two methods we have studied, and that the Austrian method is somewhat superior to either of the others. However, there is some evidence on the other side. W. W. Beatty² finds the first Italian method a trifle more rapid and a trifle less accurate than the Austrian method. Mead and Sears³ find the first Italian method a trifle better in the second grade; but the fact that the methods were taught by different teachers introduces an element of uncertainty.

Superintendent Taylor⁴ found that six years after the Austrian method was introduced into the New York City course of study only 37.6 per cent of the schools were using it.

There are two methods — often called “Austrian” or “additive” methods — that should be discussed briefly in this connection. Both of them employ the *combinations* in their additive sense; but neither regards the *whole* minuend as the sum of a given and a missing addend — and for this reason neither method can be used in balancing a ledger account.

In one of these methods 400 is changed to 39(10) and then the combinations are called thus: 8 and 2 400
are 10, 6 and 3 are 9, 1 and 2 are 3. In the other 168

¹ ROANTREE, W. F. — “The Question of Method in Subtraction”; *The Mathematics Teacher*, February, 1924.

² BEATTY, W. W. — “The Additive *versus* the Borrowing Method in Subtraction”; *Elementary School Journal*, November, 1920.

³ MEAD AND SEARS. — “Additive Subtraction and Multiplicative Division Tested”; *Journal of Educational Psychology*, May, 1916.

⁴ TAYLOR, J. S. — “Subtraction by the Addition Process”; *Elementary School Journal*, November, 1919.

method, the compensating changes of the second Italian method are made, so that the working explanation is identical with the pure Austrian method — though the thought is different. It is urged by some teachers that one of these modifications makes it easier to learn subtraction in such examples as these.

It is unwise to be dogmatic on a question of this sort. We know by observation and experience that the pure Austrian method can be used successfully in the above examples.

$$\begin{array}{r} 7\frac{1}{4} \\ 2\frac{3}{4} \\ \hline 6 \text{ yd. } 1 \text{ ft.} \\ 4 \text{ yd. } 2 \text{ ft.} \end{array}$$

It may be that the second modification described above will work better in such examples; but it is very doubtful whether any possible advantage of the “decomposition” principle could outweigh the serious disadvantage set forth early in this section.

EXERCISES

1. How should pupils solve a concrete problem in subtraction in the lower grades before the combinations have been taught? How after?

2. What must a learner know before he is prepared to begin the subtraction of large numbers?

3. How many combinations are there in subtraction? What are they?

4. Plan, or describe, a first lesson on the subtraction combinations in which 6 is the constant subtrahend, using the additive idea. Formulate five suitable problems in application of these combinations.

5. How would you have to modify the above plan if the take-away idea is to be followed?

6. Discuss drill on the combinations, showing devices for practice and how to use them.

7. Discuss in detail the types of subtraction examples given on page 65, under type II.

8. Would type II (b) require as much time to teach as would II (a)? Explain how you would associate (b) with (a). How might the examples $\begin{array}{r} 72 \\ 45 \end{array}$ $\begin{array}{r} 726 \\ 451 \end{array}$ be used in introducing type (b)?

9. Tell how you would introduce type II (c). (Most teachers would teach merely the mechanics. Try it.)

10. As on page 67 show the changes to be made in the following examples when one uses (a) the first Italian method, (b) the second Italian method.

813	5000
<u>275</u>	<u>3268</u>

11. (a) Which of the Italian methods is the more easily explained by means of objects? Explain its advantage from the viewpoint of teaching.

(b) What are the advantages of the Austrian method? The disadvantages?

12. Discuss the relative merits of the methods of subtracting described in this chapter — first in terms of the authors' arguments, then according to your own ideas. It is important that you have a *conviction*, but more important that you rise above prejudice. At the present moment, which method would you choose to teach? Why?

13. Why should a teacher understand *all* these methods?

CHAPTER IV

MULTIPLICATION

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions. — It seems incredible to us that there could have been any people of the long ago who had no practical need for finding by a shorter process than addition the number of things in a whole made up of a given number of equal groups. Yet this new process marked such a step in advance that only those nations of antiquity who stood high in civilization understood and employed it at all.

This process, which came in time to be called multiplication (*multus*, many; *plicare*, to fold), is like addition, in that two or more groups are combined to make a whole; it is different, in that the groups are equal. It may be defined as follows:

Multiplication is the process of finding the number of things in several equal groups when the number of things in each group and the number of groups are given. Or, multiplication is the process of finding the result when a number has been used a given number of times as an addend. Thus: If there are 2 cents in each of 5 groups, how many are there in the whole? We may find the whole by placing two objects in each of five groups (○○

○○ ○○ ○○ ○○), and counting the entire number to find the result. Or, we may take 2 cents 5 times as an addend ($2¢ + 2¢ + 2¢ + 2¢ + 2¢$) and find the result by adding. The number which is used as the addend, or the number to be multiplied, is called the *multiplicand*; the number by which we multiply, or the number which shows how many times the multiplicand is to be taken as an addend, is called the *multiplier*; the result obtained by multiplying is called the *product*.

Since the multiplier indicates the *number of times* the multiplicand is taken as an addend, it is always an abstract number; the multiplicand may be either abstract or concrete; if the latter, it is evident that the product always has the same denomination as the multiplicand.

The above definition presents that concept of multiplication which functions in the study of integers in arithmetic. It is true so far as it goes. Multiplication in fractions and in algebra requires an extension of this conception as expressed in the following: *Multiplication is the process of treating the multiplicand as unity must be treated to produce the multiplier*. This definition includes the one given above and also includes multiplication with fractions and with numbers having direction — positive, negative, imaginary, and complex numbers.

For example: In $\frac{3}{4}$ multiplied by $\frac{2}{3}$, to produce the multiplier, $\frac{2}{3}$, unity must be divided into 3 equal parts and 2 of them considered. Hence, $\frac{3}{4}$ multiplied by $\frac{2}{3}$ means that $\frac{3}{4}$ must be divided into 3 equal parts and 2 of them considered.

The multiplication sign. — The product of two numbers may be expressed by the sign ($\times$), which has two

readings. When the sign is preceded by the multiplicand, as in $3 \text{ lb.} \times 4$, the multiplicand 3 lb. is to be multiplied by 4; hence the expression is read *3 lb. multiplied by 4*. In the expression $4 \times 3 \text{ lb.}$, 4, the multiplier, is the number which shows how many times the 3 lb. is to be used as an addend; hence the expression is read *4 times 3 lb.* In 3×4 we cannot tell whether 3 is the multiplicand or the multiplier; hence it may be read both ways: *3 times 4* or *3 multiplied by 4*. When read as *3 times 4*, it means $4 + 4 + 4$, or 4 taken 3 times as an addend; when read *3 multiplied by 4*, it means $3 + 3 + 3 + 3$, or 3 taken 4 times as an addend.

Requisites for multiplying. — To perform the process of multiplication one should understand the fundamental methods of multiplying, the types of examples in order of progressive difficulty, and the best procedure for solving each type. To test your own mastery of multiplication see how many of the following examples¹ you can work in 3 minutes:

8246	3597	5739	2648
<u>29</u>	<u>73</u>	<u>85</u>	<u>46</u>
4268	7593	6428	8563
<u>37</u>	<u>64</u>	<u>58</u>	<u>207</u>

The median for the 8th-grade pupils of St. Louis is 5.3. How much better can you do?

Methods. — *The fundamental methods* of finding the product are counting and addition. Illustration: Find the cost of 3 pieces of candy at 2 cents each.



¹ Cleveland Survey Test — Set L.

There are one, two, three pieces (a line for each piece). There are one, two cents for this piece; one, two cents for this piece, etc. (a circle for each cent); there are one, two, three, four, five, six cents in all.

When addition is used, the work stands as below, and the final step is taken by thinking "2 cents plus 2 cents plus 2 cents are 6 cents."

2¢

2¢

2¢

The combinations. — As in addition, there are logically 45 combinations in multiplication; including the zeros there are 54: 1×0 , 2×0 , 3×0 , etc. to 9×0 ; 1×1 , 2×1 , and so on. Using the inverses as in addition there are 100, but since in multiplication we have no practical need for 0 times 0, 0 times 1, 0 times 2, etc., these combinations may be omitted, leaving 90.

There are two ways of organizing these combinations into tables: (a) according to a constant multiplicand, (b) according to a constant multiplier. To illustrate:

(a)	(b)
1 6 is 6	6 1's are 6
2 6's are 12	6 2's are 12
3 6's are 18	6 3's are 18
etc. up to	etc. up to
9 6's are 54	6 9's are 54

Each statement in the tables above may be expressed as an addition example. In table (a) in which 6 is the constant multiplicand, 6 must appear as the addend; in (b), where 6 is the multiplier, each addend must be written 6 times.

Both these arrangements should be learned.

The steps in mastering the combinations in multiplication are (1) find the results by addition, (2) memorize the results by associations, by repetitions, by motivated and selective drills, and by applications.

The following device is one of many that may be used in step 1 :

					1	2	3	4
					1	2	3	4
			6		1	2	3	4
		6	6		1	2	3	4
	6	6	6		1	2	3	4
<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>	etc.	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u> etc.

In the columns where 6 is the constant addend (multiplicand) it is easy and helpful to associate 4 6's with 2 6's (column twice as high), 6 6's with 2 6's or 3 6's; 8 6's with 2 6's or 4 6's, etc.; also each combination is easily obtained from, and associated with, the preceding combination by merely adding another 6. For these reasons it would seem wise to develop and learn a given table, first organizing it according to a constant multiplicand; but since in a given written example one is using a constant multiplier it is necessary to give much practice on a table organized according to a constant multiplier before the written work is begun. That is, in the following example we need to know 6 times 2, 6 times 9, 6 times 3, 6 times 8, etc.

$$\begin{array}{r} 478392 \\ \underline{\quad\quad 6} \end{array}$$

Increasing products by a digit. — As soon as the combinations are learned, a helpful form of practice as a preparation for multiplying larger numbers by a number of

one order is to increase each product by 1, 2, 3, etc. respectively. Thus in multiplying by 6 we may drill the pupil on calling off each of the following products increased by 1, then by 2, and so on up to 5.

9	8	6	4	2	7	1	3	5	0
<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>	<u>6</u>

The pupil says 55, 49, 37, 25, 13, 43, 7, 19, 31, 1 (increasing each by 1); 56, 50, and so on. Since in multiplying any number by 6 one never has need to carry more than 5 to the next order, it is not necessary to practice on increasing the products by any number greater than 5.

Multiplication by a number of one order. — This process may be related to the process of addition. Let us find the product of 638 multiplied by 4. The answer may be found by writing 638 four times and adding.

638		
638		
638	32√	638
638	15√	4
<u>2552</u>	25√	<u>2552</u>

Instead of writing 638 four times, we may write it once and below it the number of times it is to be used; instead of finding the result by adding we may find it through our knowledge of the multiplication combinations by multiplying the number in each order separately. In the *complete explanation* each step in the process is shown: 4 times 8 units are 32 units or 3 tens and 2 units; we write the 2 in units' place and carry the 3 to add to the tens. 4 times 3 tens are 12 tens; 12 tens and 3 tens are 15 tens or 1 hundred and 5 tens; we write the 5 in tens' place, etc.

Working explanation. — It is desirable to form the habit of saying as little as possible when one works. It is just as easy first to form the habit of saying 32 at once as to say 4 times 8 are 32. In early stages the pupil may say 32; 12, 15; 24, 25. Later he should be able to call at once the result of each product after it has been increased, thus: 32, 15, 25. It follows that facility in this process depends upon the ability to call at sight the results of combinations increased by any number which might be carried. This exercise is described on page 81.

Multiplication by powers of 10. — Since 10 times a unit equals a ten, 10 times a ten equals a hundred, 10 times a hundred equals a thousand, and so on, *we may multiply any number by 10 by causing each figure of that number to occupy the next higher order.* Thus, 238 or 2 hundreds 3 tens 8 units multiplied by 10 equals 2 thousands 3 hundreds 8 tens or 2380:

Th	H	T	U	
	2	3	8	
2	3	8		= 10 times 238

The common rule "Annex a zero to the number to multiply it by 10" is invalid when the multiplicand is a decimal; neither is it used in an ordinary multiplication example of two or more orders. Thus in multiplying 234 by 14 in the ordinary way, the second partial product, which is really 10 times the number, does not have the zero annexed:

$$\begin{array}{r}
 234 \\
 14 \\
 \hline
 936 = 4 \text{ times } 234 \\
 234 = 10 \text{ times } 234 \\
 \hline
 3276 = 14 \text{ times } 234
 \end{array}$$

The rule "To multiply a number by 10, move each digit of the multiplicand one place to the left," applies to integers, to decimals, and to the method employed in ordinary multiplication.

To multiply by 100 move each figure of the multiplicand two places to the left ; by 1000, three places to the left, etc. This is true because to multiply by 100 is to multiply by 10 twice ; by 1000, is to multiply by 10 three times, etc.

Multiplication by a number of two orders. — Let us consider the multiplication of 385 by 34. To multiply by 34 is to multiply by 30 and by 4 and add the results (since $34 = 30 + 4$).

$$\begin{array}{rcl}
 & 386 & \\
 34 & 34 = 30 + 4 & \\
 \hline
 1544 & = 4 \text{ } 386\text{'s} & \\
 1158 & = 30 \text{ } 386\text{'s} & \\
 \hline
 13124 & = 34 \text{ } 386\text{'s} &
 \end{array}$$

We first multiply by 4, the number in units' order, as in multiplication by a number of one order, obtaining 1544. We next multiply by 3, the number in tens' place, and this result by 10 (as 30 is 3×10), by moving each digit one place to the left ; this gives 11580, but it is not necessary to write the 0, as explained on page 82, since each order has its place value from the preceding product. Adding the partial products, we obtain 13124.

Multiplication by a number of more than two orders. — Extending the procedure of the preceding example and applying the rule for multiplying by 100, 1000, etc., we discover that the general rule for multiplying is as follows: Multiply by the number in units' order, mul-

tiply by the number in tens' order and move each digit one place to the left, multiply by the number in hundreds' order and move each digit two places to the left, etc. ; add the partial products. Illustrations :

$$\begin{array}{r}
 568 \\
 234 \\
 \hline
 2272 = 4 \text{ 568's} \\
 1704 = 30 \text{ 568's} \\
 1136 = 200 \text{ 568's} \\
 \hline
 132912 = 234 \text{ 568's}
 \end{array}$$

$$\begin{array}{r}
 7236 \\
 4008 \\
 \hline
 57888 = 8 \text{ 7236's} \\
 28944 = 4000 \text{ 7236's} \\
 \hline
 29001888 = 4008 \text{ 7236's}
 \end{array}$$

Multiplication by multiples of 10, 100, etc. — Let us consider the multiplication of 236000 by 340. Since $236000 = 236 \times 1000$ and $340 = 34 \times 10$, their product is $236 \times 1000 \times 34 \times 10$. The product will be the same in whatever order the numbers are multiplied. We may therefore multiply 236 by 34 and this result by 1000×10 . Hence *we may multiply without regard to the ciphers and then annex as many zeros to the product as we have disregarded.* The work should be arranged as follows :

$$\begin{array}{r}
 236000 \\
 340 \\
 \hline
 944 \\
 708 \\
 \hline
 80240000
 \end{array}$$

Care to keep the figures in strict column arrangement will result in the avoidance of mistakes.

Checks. — There are two common methods of checking multiplication : (1) by going through the work a second time ; (2) by multiplying again, having interchanged the terms. Method (1) takes less time, but method (2) is

excellent for persons who are very inaccurate and is particularly suited to the third and fourth years, where much practice is needed to master the process of multiplication. It depends upon the foundation principle of multiplication, known as the *commutative law*: *The product is the same in whatever order the numbers are taken; i.e., interchanging multiplicand and multiplier does not affect the product.* Proof: Take four rows of three circles each,



and we have 4 3's; taking 3 vertical rows of 4 circles each, we have 3 4's. The entire number of circles is constant; hence the result of 4 3's is the same as the result of 3 4's. In the same way the rule may be proved true no matter what numbers are chosen for the terms, and hence the rule is general.

Multiplication may also be proved by the excess of nines.

We may discover the rule inductively. Let us take the examples 564 multiplied by 26 and 427 multiplied by 63.

Ex. of 9's in	564 is 6
Ex. of 9's in	26 is 8
	3384 48
	1128
Ex. of 9's in	14664 is 3
Ex. of 9's in	48 is 3

Ex. of 9's in	427 is 4
Ex. of 9's in	63 is 0
	1281 0
	2562
Ex. of 9's in	26901 is 0
Ex. of 9's in	0 is 0

In each of the above cases it is seen that the excess of nines in the product is the same as the excess of nines in the product of the excesses of nines in the multiplicand and multiplier. Hence the rule is probably true: *To prove multiplication by 9's, multiply the excess of 9's in the multiplicand by the excess of 9's of the multiplier; the excess of 9's of this result should equal the excess of 9's in the original product.*¹

The work should appear thus :

$$\begin{array}{r}
 438 \quad 6 \\
 \quad 67 \quad 4 \\
 \hline
 3066 \quad 24 \\
 2628 \\
 \hline
 29346 \quad 6\sqrt{}
 \end{array}$$

Mental multiplication. — Mental multiplication of a number of two orders by a number of one order is of high practical value. The method is to multiply first the tens, then the units, and then add the results. *E.g.*, multiply 36 by 7. $7 \times 30 = 210$; $7 \times 6 = 42$; $210 + 42 = 252$. The working explanation is: 210, 42, 252.

HISTORICAL NOTES

Earliest methods. — As previously stated, only those nations among the ancients who stood high in civilization

¹ *Deductive Proof.* — Let F^1 and F^2 be the factors, M^1 and M^2 the respective multiples of 9 in the multiplicand and multiplier, and E^1 and E^2 the respective excesses of nines in the factors, and P the product. Then, —

$$\begin{array}{l}
 F^1 = M^1 + E^1 \\
 F^2 = M^2 + E^2
 \end{array}$$

$P = M^1M^2 + M^2E^1 + M^1E^2 + E^1E^2$, since the products of equal numbers are equal. Ex. of nines in $P =$ Ex. of nines in $(M^1M^2 + M^2E^1 + M^1E^2 + E^1E^2)$. Hence Ex. of nines in $P =$ Ex. of nines in E^1E^2 , because the first three terms in the expression within the parentheses above are multiples of nine and have no excesses. Hence, the following principle:

The excess of nines in the product equals the excess of nines in the product of the excesses of nines in the multiplicand and the multiplier.

had need for a process so advanced as multiplication. Multiplication tables, the bugbear of many a child, were in use centuries before Christ, among the ancient Babylonians.¹ The systems of notation in use among the ancient Egyptians, Romans, and Greeks did not lend themselves to the fundamental combinations that are possible with the Hindu numerals, although the Greeks had a form of multiplication table given in an early textbook on arithmetic written by Nicomachus about 100 A.D.² Column multiplication was performed by the Greeks and Romans by repeated additions on the abacus, supplemented by a table of multiplications which was learned by heart.³

Methods after the introduction of Hindu notation. — The Hindus and Arabs, who used our present system of notation, could profit by the use of tables and did so. They also invented several different methods for multiplying larger numbers, illustrations of which may be found in most histories of mathematics. After the introduction of the Hindu notation into Europe, multiplication tables came into vogue and were printed in the early arithmetics in varying forms. The form in Fig. 10, page 88, was used by Widman,⁴ who published an arithmetic in Leipzig in 1489.⁵

One of the most interesting forms used for the multiplication tables was invented by Napier about 1617, known as "Napier's Rods." The tables were written on thin, rectangular strips of bone, metal, or cardboard.⁶

¹ SMITH, D. E. — *History of Mathematics*, p. 40.

² *Ibid.*, p. 129.

³ BALL, W. W. R. — *Short History of Mathematics*, p. 132.

⁴ SMITH, D. E. — *Number Stories of Long Ago*, p. 68.

⁵ BALL, W. W. R. — *Short History of Mathematics*, p. 212.

⁶ *Ibid.*, p. 196.

1									
1	2								
2	4	3							
3	6	9	4						
4	8	12	16	5					
5	10	15	20	25	6				
6	12	18	24	30	36	7			
7	14	21	28	35	42	49	8		
8	16	24	32	40	48	56	64	9	
9	18	27	36	45	54	63	72	81	

FIGURE 10

To find the product of two numbers find that square which is in the same vertical column as one of the numbers and in the same horizontal line as the other; the number contained therein is the product required.

Many different methods were in vogue among the early Italians when arithmetics were first printed. Some of these are shown in the earliest printed arithmetic known as the Treviso Arithmetic,¹ which appeared in 1478. Pacioli, in his arithmetic printed in 1494, gives several

	3	4	2	
4	1	1	8	8
6	1	2	1	8
8	2	3	1	4
	2	9	5	

Answer: 295,488

FIGURE 11

different methods. An interesting form is the "latticed multiplication,"² by which the multiplication of 342 by 864 would appear as in Figure 11.

EXERCISES

1. Write the columns for finding the results of the following by addition: (a) 8 times 4, (b) 8 multiplied by 4, (c) 8×4 lb., (d) $\$3 \times 5$, (e) 4 8's, (g) 8 4's, (h) 6×5 . Discuss the reading of the sign in (c), (d), and (h).

2. Regarding the combinations in Exercise 1, page 36, as multiplication combinations, (a) call the products and record your time in seconds; (b) write the results as directed in Exercise 2, same page;

¹ SMITH, D. E. — *History of Mathematics*, p. 248

² CAJORI. — *A History of Elementary Mathematics*, p. 143.

(c) try calling the results rhythmically, using a pendulum as suggested on page 45.

3. Write the table in which 7 is the constant multiplier, using the form of expression employed in 1 (a); the form employed in 1 (b); in 1 (e). Which of these forms of expression is most desirable for use in written multiplication? Why?

4. If a table is written in the form shown in 1 (h), can you tell which factor is the multiplier?

5. Write the products to not more than 100 in which 13 is a factor; in which 14 is a factor; in which 15 is a factor; and so on to 50×2 . Check the products with which you are so familiar that you can call the results instantly. Master the remaining combinations so that you can call their results miscellaneously at sight. Be prepared to tell how you proceeded and report the number of combinations mastered, the time spent in learning them, and how this time was distributed.

6. Time yourself on the following: Think the combinations of exercise 5 — 1 times 13, 2 times 13, etc. — writing the results as indicated here. Did you write the results from memory, or did you find them by mental multiplication?

13	14
26	28
.	.
.	.
.	.
.	.
91	98

7. Multiply and check by casting out nines:

- | | |
|---------------------|--------------------|
| (a) 46894 by 287 | (e) 877968 by 4812 |
| (b) 96738 by 568 | (f) 3680 by 9700 |
| (c) 828769 by 927 | (g) 76800 by 47 |
| (d) 5867926 by 5607 | (h) 8342 by 7800 |

8. In 7 (a) the answer may be obtained, or checked, by multiplying first by 7, then multiplying this partial product by 40 (since 280 is 40×7), and adding these results. Thus:

$$\begin{array}{r}
 46894 \\
 287 \\
 \hline
 328258 = 7 \times 46894 \\
 1313032 = 40 \times \text{1st partial product} = 280 \times 46894 \\
 \hline
 13458578 = 287 \times 46894
 \end{array}$$

90 AN ARITHMETIC FOR TEACHERS

Which of the other examples may be similarly worked? Use this method as a check on these examples.

9. Extend and find the amount of the following bill: ¹

NASHUA, N. H., Nov. 25, 1924

MR. STEPHEN SCOTT, CONCORD, N. H.

Bought of SILVER & YOUNG

	13 Linen shades	@ \$1.88		
	26 sq. yd. linoleum	@ 2.35		
	38 yd. carpet	@ 2.78		
	8 Dining chairs	@ 18.75		
	6 Porch chairs	@ 9.85		

10. Fill in the blanks:

CORN PRODUCTION ²

STATE	NUMBER OF THOUSAND ACRES	AVERAGE YIELD PER ACRE	TOTAL YIELD	AVERAGE PRICE PER BU.	TOTAL VALUE
New York . .	512	38.6		.70	
New Jersey .	273	38.0		.68	
Pennsylvania .	1499	42.5		.63	
North Carolina	2808	18.2		.83	
Georgia . . .	3910	13.8		.85	
Florida . . .	655	13.0		.79	
Totals					

11. Cut 9 strips of stiff paper $4\frac{1}{2}$ inches by $\frac{1}{2}$ inch and mark off each strip into 9 squares. On these strips write the products of the nine multiplication tables; the products in the table of 3's is shown in (a). These are the famous "Napier's Rods." In (b) is illustrated the use of these rods in multiplication — multiplying 674 by 8. Count down in (b) to the eighth horizontal row of squares. To find the desired

¹ See pages 341–343 for a brief discussion of bills.

² FINNEY AND BROWN. — *Modern Business Arithmetic*, p. 27; Henry Holt and Company.

product add diagonally : 2 units + (6 + 3) tens + (8 + 5) hundreds + 4 thousands = 5392.

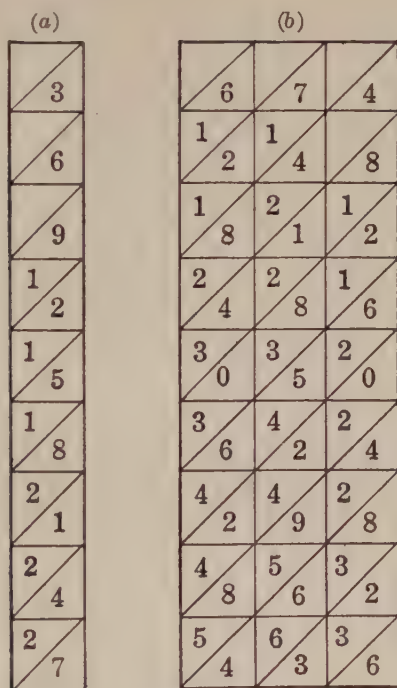


FIGURE 12

From (b) find the following products: 2×674 , 5×674 , 7×674 , 34×674 , 57×674 .

Using the appropriate rods, find the following products: 7×275 , 34×275 , 6×5486 , 437×874 .

Find the product of the first five examples by the latticed multiplication, and record the time. Repeat the experiment with the next five examples. What is the effect of the practice?

$$6 \times 274, 4 \times 583, 8 \times 856, 3 \times 437, 7 \times 673$$

$$4 \times 397, 5 \times 643, 6 \times 742, 7 \times 348, 8 \times 637$$

92 AN ARITHMETIC FOR TEACHERS

12. Copy and complete the following Time Sheet.

NAME	M	T	W	Th.	F	S	TOTAL TIME	WAGE PER HR.	WEEK'S WAGE
H. C. Hall	8	6	7	8	5	8		.85	
Geo. Hood	9	7	6	8	4	9		1.15	
L. M. Dart	5	7	9	8	7	3		1.05	
S. J. Kling	8	9	6	7	8	4		.95	
T. Dobson	8	7	9	6	8	5		.75	
J. H. Mason	9	7	8	6	7	5		.80	
R. C. Ford	7	7	8	8	9	6		.85	
Totals									

13. Copy and fill in the blanks by finding how many pieces of each denomination must be drawn from the bank to place in an envelope the pay for each person in Exercise 12. Check the results with results in Exercise 12.

NAME	\$10	\$5	\$1	\$.50	\$.25	\$.10	\$.05	\$.01	TOTAL
Hall	—	—	—	—	—	—	—	—	—
Hood	—	—	—	—	—	—	—	—	—
Dart	—	—	—	—	—	—	—	—	—
Kling	—	—	—	—	—	—	—	—	—
Dobson	—	—	—	—	—	—	—	—	—
Mason	—	—	—	—	—	—	—	—	—
Ford	—	—	—	—	—	—	—	—	—
Totals	—	—	—	—	—	—	—	—	—

14. State the squares from 1^2 through 25^2 . Thus, $1^2 = 1$, $2^2 = 4$,

...

15. State the cubes from 1^3 through 9^3 . Thus, $1^3 = 1$, $2^3 = 8$, . . .

16. State a problem involving multiplication and tell which number is the logical multiplier.
17. Illustrate "latticed multiplication" in the example 3468×64 .
18. Multiply mentally:

(a)	(b)	(c)	(d)
7×43	3×68	5×960	3×6400
8×64	9×84	7×830	8×2700
4×73	6×37	6×470	4×3900

19. Find the results:

(a)	$1 \times 9 + 2 = //$	(b)	$9 \times 9 + 7 = 88$
	$12 \times 9 + 3 = '//'$		$98 \times 9 + 6 = 888$
	$123 \times 9 + 4 = '///'$		$987 \times 9 + 5 = 8888$
	$1234 \times 9 + 5 = '////'$		$9876 \times 9 + 4 = 88888$
	$12345 \times 9 + 6 =$		$98765 \times 9 + 3 = 888888$
	$123456 \times 9 + 7 =$		$987654 \times 9 + 2 = 8888888$
	$1234567 \times 9 + 8 =$		$9876543 \times 9 + 1 = 88888888$
	$12345678 \times 9 + 9 =$		$98765432 \times 9 + 0 = 888888888$

20. Find the product of the numbers in every row, diagonal, and column in Fig. 13. (This is called a geometrical magic square.)¹

METHODS OF TEACHING

The combinations. — The question as to the order² in which the combinations should be learned is as yet a matter of opinion. A very few educators oppose the learning of tables. Of those who would teach tables a few believe the order quite un-

256	2	64
8	32	128
16	512	4

FIGURE 13

¹ WHITE, W. F. — *Scrap Book of Mathematics*, p. 186; Open Court Publishing Company.

² JESSUP, W. A. AND COFFMAN, L. D. — *The Supervision of Arithmetic*, p. 79; The Macmillan Company.

THORNDIKE, E. L. — *Psychology of Arithmetic*, p. 141; The Macmillan Company.

STARCH, D. — *Educational Psychology*, p. 406; The Macmillan Company.

important; many would follow the traditional order, the 1's, 2's, 3's, etc.; nearly as many others say the 2's, 4's, 5's, 10's, but split badly as to the order for the other tables. It is quite possible there is a best order and that it may be discovered by elaborate experimentation. At any rate, we shall do well as a matter of convenience to teach the tables as such. As to sequence, good arguments¹ may be made for any of the following:

- (1) 1's, 2's, 3's, etc.
- (2) 5's, 2's, 4's, 8's, 3's, 6's, 9's, 7's.
- (3) 5's, 2's, 4's, 3's, 6's, 9's, 8's, 7's.
- (4) 2's, 4's, 5's, 3's, 6's, 9's, 8's, 7's.
- (5) 2's, 1's, 10's, 11's, 9's, 5's, 0's, 3's, 4's, 6's, 7's, 8's, 12's.

One of the best arguments for teaching them in consecutive order is that the number of new combinations (those whose results have not been learned in a preceding table) in each table is one less than in the preceding table. In taking up any new table after the 1's, those combinations previously learned should be treated as review; hence the major part of the drill should be devoted to the new combinations.

As has been previously noted the organization according to a constant multiplicand takes advantage of useful associations, makes the meaning of the facts very clear, and hence facilitates the first steps in learning. It is a mistake to teach thoroughly one combination at a time, for this misses the advantages just named and is very monotonous. Give the children a chance to grasp the scheme of a table as a whole and then take up miscella-

¹ GILDEMEISTER, THEDA. — *The Multiplication Tables*; A. Flanagan Company, N. Y., 1905.

neous drill on it for short intervals daily. Drills on the tables should precede the miscellaneous drills for the same reasons that obtain in memorizing tables in addition — a pupil has a means at his command for recalling a combination that he may fail upon in miscellaneous drills, without having to find the result again by the fundamental method of adding. It is a common practice to prepare pupils for mastery of the combinations organized into tables according to a constant multiplicand by counting numbers of one order. The following steps constitute a sound procedure in teaching the 3's:

1. Give preparatory drill on counting by 3's to 30.
2. If the traditional sequence of tables is chosen, review the table as far as it is known: 1 3 is 3, 2 3's are 6.
3. Derive the results of the new combinations by column addition for the table in which 3 is the constant multiplicand (see page 80).
4. Drill on repeating the entire table.
5. Give miscellaneous drill on the combinations. (For procedure and devices see suggestions for drill on addition combinations, page 42.)
6. Bring out the associations between the various combinations (see page 80).
7. Derive the combinations of the table in which 3 is the constant multiplier. In finding the results, column addition must be used for a time at least; by some teachers it is thought advisable to use addition until the children see that the product of two numbers is the same whichever is taken as the multiplicand, and then to depend upon this principle for subsequent new combinations.
8. Drill on the entire table, 3 1's are 3, 3 2's are 6, etc.
9. Give miscellaneous drill on the combinations derived in 7.
10. Apply the combinations in written examples without carrying,
as: $\begin{array}{r} 312 \\ 3 \end{array}$ $\begin{array}{r} 63 \\ 3 \end{array}$ and so on.

11. Give miscellaneous drill on all combinations in which 3 appears either as multiplicand or multiplier, as: $\begin{array}{r} 3 \\ 6 \end{array}$ $\begin{array}{r} 4 \\ 3 \end{array}$ $\begin{array}{r} 9 \\ 3 \end{array}$ $\begin{array}{r} 3 \\ 8 \end{array}$ and so on.

12. Increase products in the table of "3 times," by 1, or by 2 (see page 80). Thus, add 1 to each result :

2	6	1	0	4	8	9	7	5	3
<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>	<u>3</u>

At first the pupil will slowly think : 6, 7 ; 18, 19 ; and so on. But later he will give the final results only : 7, 19, 4, etc.

13. Give oral practice on the working explanation (see page 82)

of examples like : $\begin{array}{r} 45 \\ 3 \end{array}$ $\begin{array}{r} 637 \\ 3 \end{array}$ $\begin{array}{r} 829 \\ 3 \end{array}$ $\begin{array}{r} 504 \\ 3 \end{array}$ The more capable

pupils will be able, after properly organized drill, to call the results practically at sight ; thus, 15, 13 ; 21, 11, 19 ; etc. But none of the children will be able to respond in this manner at the very start. If the children are well founded in the combinations, they will be able, first, to say : 15, 12, 13 ; 21, 9, 11, 18, 19 ; and so on. Later they can give the shorter form.

14. Written multiplication with carrying. As in all written work, the first steps in written multiplication should be taken under the eye of the teacher, and the children should be given much practice in which the working explanation and the motor responses of writing the figures in the answer are properly combined and coördinated to form the desired working habit.

Written multiplication rationalized. — The steps given above have to do with the development of a serviceable group of habits. It is a moot question whether rationalizing the process is of assistance in establishing the desired habits. If it is, the rationalizing should come in between steps 12 and 13 ; if it is not a help, it should be deferred until the habit is established, and this means with all, except possibly the more intellectual pupils, it will be postponed indefinitely.

Thorndike repeatedly urges the point that children as a rule are not interested in a merely theoretical explanation

of an abstract process at the time it is being learned, but that they are entitled to know the *meaning* of a process and to understand that it helps one to get desired results.¹ The explanations given in this chapter are of three sorts: those that reveal the meaning of the process (multiplication as a short addition), those that show the principles applied (complete explanation), and those that describe the mechanics of the process (working explanation).

Probably the best approach to an understanding of written multiplication is through its use in the solution of problems. Let us illustrate this in the case of the more important types of examples.

1. *Short multiplication.*

Problem: In our school there are 4 third-year classrooms with 48 seats in each room. How many seats in these 4 rooms? The pupils can get the answer by adding as in (a).

(a)	(b)
48	
48 32 ✓	48
48 19 ✓	4
48	192
192	

In explaining method (b) point out that units' column in (a) is 4 8's, which can be handled as a multiplication in (b). We write 2 in the answer and carry 3 tens just as we did in (a). In (a) we take 3 first and then add the 4's, but in (b) we take the 4 4's first and then add 3.

¹ THORNDIKE, E. L. — *Psychology of Arithmetic*, pp. 23, 60, 115, 116, 193, 194; The Macmillan Company.

2. *Multiplying by a number in the "teens."*

Problem: In the primary department of our school there are 14 rooms with 48 seats in each room. How many children could we take care of in our primary department?

The children may first think of adding, but this is much too hard work. The best way is to find how many seats there are in 4 rooms, then observe that there are 10 more rooms, which requires multiplying by 10.

$$\begin{array}{r}
 48 \\
 \underline{14} \\
 192 \text{ seats in 4 rooms} \\
 48 \text{ seats in 10 rooms} \\
 \hline
 672 \text{ seats in 14 rooms}
 \end{array}$$

From a review of the facts, 10 2's are 20, 10 3's are 30, and so on, the pupils may be led to infer that 10 times 48 are 480. (Or the rule on page 82 may be developed.) We have found there are 480 seats in 10 rooms. How shall we find how many seats there are in all? The pupils add and declare their answer. It is advisable to show that one need not write the 0 in 480 because the sum will be correct if he writes the 8 in tens' place.

3. *Multiplying by a number of two figures.*

Problem: In our school there are 34 rooms with 48 seats in each room. How many children could we seat in our school?

The same arrangement as in Type 2 would be used, and the reasoning is the same except for the new step — the number of seats in 30 rooms is 3 times the number in 10 rooms.

Examples with three-figure multipliers may be similarly

taught. Multipliers like 406 are often permitted to give needless trouble. Some teachers, instead of appealing to the common sense of children, resort to some mechanical device. The forms (a) and (b) shown below, especially (a), are not infrequently used by high school graduates :

(a)	(b)
374	374
406	406
<u>2244</u>	<u>2244</u>
14960	000
<u>151844</u>	1496
	<u>151844</u>

A child of ordinary intelligence can associate 374 multiplied by 406 with 374 multiplied by 426. Ask him to work example (c) and interpret each partial product ; then give him example (d) and ask him what partial products in (c) are needed in (d).

(c)	(d)
374	374
426	406
<u>2244</u> = 6 times 374	<u>2244</u> = 6 times 374
748 = 20 times 374	1496 = 400 times 374
1496 = 400 times 374	<u>151844</u> = 406 times 374
<u>159324</u> = 426 times 374	

Show that 20 times 374 must not appear in (d) and that the position of the last partial product is exactly the same in (d) as in (c), viz., the right-hand figure, 6, stands in the column two places to the left of units.

That there are some children who are unable to profit by the explanations here suggested is no argument against the explanations — or the children. If such explanations

give satisfaction to the upper quarter of the class, they are distinctly worth while. The time which the explanation takes gives the slower children a better opportunity to master the mechanics, and any points which they are not ready to understand will pass unheeded. If the explanation centers the attention upon getting the correct answer to a meaningful problem, each child can be trusted to get from it what it has for him. It would be a mistake to *insist* upon the children understanding the argument — still worse to ask them to *reproduce* the explanation ; but the worst mistake is to permit naturally thoughtful children to grow up with the idea that arithmetic is a meaningless aggregation of sleight-of-hand performances, requiring nothing higher than rote memory and mechanical skill. Modern psychology advocates no such practice.¹

Mental multiplication (p. 86). — Multiplication of a two-place number by a number of one figure finds immediate application in long division where it is useful in determining whether a trial quotient figure (p. 113) will do. Moreover, there are countless occasions in everyday shopping or marketing for the practical use of this ability ; hence such problems should be used for motivating the work. As in mental addition and subtraction with two-order numbers, the work should begin with multiples of 10 ; it is easy to associate 4 times 30 with 4 times 3, etc. The work with other multiplicands should first be presented to the eye, and the full explanation very carefully drilled. Later the work should be carried on without the numbers in sight and the working explanation omitted. It is important that the children should become independent of the

¹ THORNDIKE, E. L. — *Psychology of Arithmetic*, p. 115.

written numbers, for in shopping and marketing problems the numbers will generally have to be carried in mind.

The device shown below presents sixty-four different examples, and is so made that practically all the combinations are covered. If the children understand that every number in the right-hand column is to be multiplied by 2, then by 3, and so on, it will not be necessary for the teacher to announce or point to the examples.

The second time through, it can be understood that 31 is to be multiplied by every number in the left-hand column, then 52, and so on.

Higher tables. — The reason for extending the multiplication combinations through the *twelves* is that there are frequent occasions in practical life for changing dozens to units, gross to dozens, feet to inches, and it is a convenience to carry on these computations mentally. The arguments that these combinations enable the children to use short multiplication and division by 11 and 12 would apply with equal force to 13, 14, and so on indefinitely. Memorizing the tables of 13, 14, etc. up to 19 is an excellent preparation for long-division examples of the type, $1468 \overline{)82537}$. Instead of the trial division of 8 by 1, we may use 82 divided by 14 and the exact quotient figure is thus found on first trial (see page 114).

2	}	X	{	31
3				52
4				43
5				67
6				84
7				28
8				76
9				95

FIGURE 14

EXERCISES

1. What are the arguments for and against teaching the multiplication combinations in tables?

2. Discuss the order in which the multiplication tables may be studied. Which of these orders would you choose to follow in teaching?

3. How many pints in 3 quarts? (a) Solve by counting, showing the board work; (b) solve by adding, showing the board work.

4. Prepare a detailed plan for teaching the combinations in which 4 is the constant multiplicand.

5. In the study of the above table it is well for the children to see that the product of 4 4's is twice as great as the product of 2 4's, etc. List all the similar associations occurring in this table. What is the value of noting these relationships?

6. Prepare a detailed plan for teaching the combinations in which 6 is the constant multiplier.

7. In the study of the 6 *times* table is it possible to make associations similar to those made in the study of the 4 *times* table? Discuss.

8. Describe, or show, several devices for miscellaneous drill on the combinations with 6. Discuss their uses, and state which should be used with greatest frequency.

9. When and how would you teach the zero combinations?

10. Show the board work you would employ in giving a five-minute drill on increasing the products in the 4 *times* table. Plan in detail how you would conduct the drill.

11. Assuming that sufficient drill of the kind considered in the last exercise has been given, and the children are ready to begin written multiplication by 4, tell, or illustrate, how you would start the children working written multiplication examples at the blackboard.

12. Would you expect all children to understand the *whys* of written multiplication as developed in this chapter? Would you require any of them to give formal explanation of these processes? Taking each type of example separately, frame some questions which would test a child's understanding of the process, and give what you would consider good *child's* answers to these questions.

13. Give the arguments for and against the plan of teaching written multiplication through sheer imitation.

14. Describe the common methods of checking multiplication. Which of these methods would you have the children use before the check by 9's has been learned? Why?

15. It has been found feasible to teach the check by 9's as early

as the fifth year. If you decided to teach this method of checking in the fifth year, would you use the methods suggested on page 85, or would you have the children learn by mere imitation?

16. From the child's point of view what is the advantage of learning the check by 9's?

17. Find the cost of 35 bicycles at 47 dollars each. Making use of this problem, give a teaching explanation of the process of multiplying by a number of two orders.

CHAPTER V

DIVISION

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions. — The history of the development of man's thought processes as revealed in his gradually increasing knowledge of number relations is as interesting as the story of the development of plant and animal life. From the earliest known process of counting there gradually emerged in succession the processes of shortening the counting by *adding*, of inverting addition by *subtracting*, of shortening the addition by *multiplying*, and of reversing multiplication. The process that arises when multiplication is reversed is known as *division*. It has two aspects. Suppose we have 4 bags of flour each containing 5 lb.; in the whole we have 20 lb. of flour.

1. If the 4 is unknown, our question becomes: By what number must 5 lb. be multiplied to produce 20 lb.? That is, the multiplier is to be found, $5 \text{ lb.} \times ? = 20 \text{ lb.}$ Or, the question may take this form: How many times must 5 lb. be used as an addend to produce 20 lb.? $5 \text{ lb.} + 5 \text{ lb.} + \dots = 20 \text{ lb.}$ Or we may ask the question thus: Into how many equal parts must a whole which contains 20 lb. be divided so as to have 5 lb. in each part? ○○○○○ ○○○○○ ○○○○○ ○○○○○. The pro-

cess arising from these questions we will call *quotition*¹ (*quot* is the Latin for *how many*). It is also known as the *measurement* aspect of division.

2. If the 5 lb. is wanting, we are asked to find: What number multiplied by 4 will give 20 lb.? That is, the multiplicand is to be found. $? \text{ lb.} \times 4 = 20 \text{ lb.}$?

Or, the question may be related to addition, What number used 4 times as an addend will produce 20 lb.? $? \text{ lb.} + ? \text{ lb.} + ? \text{ lb.} + ? \text{ lb.} = 20 \text{ lb.}$? Or we may ask: How many lb. are there in each of the 4 equal parts into which 20 lb. may be divided? ○○○○○ ○○○○○ ○○○○○ ○○○○○. The process arising from these questions is known as *partition*. A threefold definition, corresponding to the three questions stated above, may be given for each process.

Quotition or *measurement division* is the process of finding how many equal groups there are in a whole when the number of units in the whole and the number of units in each of the equal groups are given. Or, *quotition* is the process of finding how many times a given number must be used as an addend to produce a given sum. Or, *quotition* is the process of finding the multiplier when the product and multiplicand are given.

Partition is the process of finding the number of units in each equal group when the number of units in the whole and the number of equal groups are given. Or, *partition* is the process of finding what number used a given number of times as an addend will produce a given sum. Or, *partition* is the process of finding the multiplicand when the product and the multiplier are given.

¹ M. A. Bailey originated this term. See his *Teaching Arithmetic*, p. 76.

Since the general term *division* includes both of these processes, its definition must apply to both.

Division is the process of finding one of two numbers when their product and the other number are given. The given product is called the *dividend*, or the number to be divided; the other given term is called the *divisor* or the number by which we divide, and the required term is called the *quotient* or the result obtained by dividing.

The division of two numbers may be expressed by the sign ($\div$), which is read *divided by*. Thus: 15 lb. $\div$ 5 means that 15 lb. is to be divided into 5 equal parts, and is read 15 lb. *divided by* 5. The division of 15 lb. by 5 may also be written $\frac{1}{5}$ of 15 lb., which is read *one-fifth of 15 lb.* 15 lb. $\div$ 5 lb. means that 15 lb. is to be divided into groups each containing 5 lb.; it is read 15 lb. *divided by* 5 lb. Division may also be expressed thus: 3 lb.)15 lb., which is read 15 lb. *divided by* 3 lb. A fraction is also an indicated division, as $\frac{15 \text{ lb.}}{5 \text{ lb.}}$, which is read 15 lb. *divided by* 5 lb. (see page 157).

Remainders. — Often the division is not exact, as in the following problems: (1) At \$3 each how many shirts can be bought for \$17? 5 shirts, and there remain \$2 which are not divided at all. (2) A class of 181 students are to be divided into 8 (as nearly as possible) equal sections. How many in each section? Each section must contain 22 students, and the remaining 5 students must be arbitrarily assigned, each to a section. In these problems \$2 and 5 students are called the *remainders*. A *remainder*¹ in division is an undivided portion of the dividend. It

¹ An appropriate word since it denotes the result of a subtraction process.

may be expressed by the word *with* followed by the number which remains. Thus in dividing 181 by 8, the result is 22 *with* 5. Or we may say, "The quotient is 22; the remainder is 5."

Different forms of expressing the result. — It is necessary to keep clearly in mind the distinction between a remainder and the fractional part of a *complete* quotient. The latter is often incorrectly called the remainder. Thus, in $58 \div 9$ the answer may be written in either of two ways, as a *quotient with remainder* or as a *complete quotient*.

$$\begin{array}{r} (a) \\ 9 \overline{)58} \\ \underline{54} \\ 4 \end{array}$$

6, with 4

$$\begin{array}{r} (b) \\ 9 \overline{)58} \\ \underline{54} \\ 4 \end{array}$$

$6\frac{4}{9}$

In (a), 6 is the integral quotient and 4 is the remainder; in (b), $6\frac{4}{9}$ is the complete quotient of which 6 is the integral part and $\frac{4}{9}$ is the fractional part.

Practical problems in division in cases where the division is not exact are of three sorts: those requiring (a) the integral quotient only (even though there be a remainder, some situations may arise in which no interest attaches to the remainder), (b) those requiring the quotient with remainder, (c) those requiring the complete quotient.

Illustration: (a) How many dresses each requiring 5 yards can be cut from a bolt containing 32 yards? The answer required is 6 dresses (integral quotient; the remainder is disregarded). (b) A bought chairs at \$16 apiece. How many can he buy for \$100 and what would he have left? Here the answer is 6 chairs with \$4 remaining (quotient and remainder). (c) A strip of ribbon one yard in length is to be cut up into 10 badges of equal length. What is the length of each? *Ans.*, $3\frac{6}{10}$ in. (complete quotient).

Requisites for teaching division. — The teacher's understanding of the process of division should include a knowledge of (1) the fundamental methods upon which it depends, (2) the types of examples in the order of progressive difficulty, (3) the best procedure for solving each type, (4) a fair degree of ability in performing the process.

To test your own ability in division see how many of the following examples ¹ you can work accurately in three minutes:

$$\begin{array}{cccc} 67 \overline{)32763} & 48 \overline{)28464} & 97 \overline{)36084} & 59 \overline{)29382} \\ 78 \overline{)69888} & 88 \overline{)34496} & 69 \overline{)40296} & 38 \overline{)26562} \end{array}$$

The median for the 8th-grade pupils of St. Louis is 2.7; a teacher should be able to work from five to six in this length of time. Where do you stand?

Fundamental methods. — The fundamental method of division is subtraction. In partition we may find what number subtracted a given number of times from the dividend will give a remainder less than the number thus subtracted.

Illustration: $15 \text{ lb.} \div 3 = ?$ Find how many pounds may be subtracted 3 times. Try 1 lb., 2 lb., etc., up to 5 lb. $15 \text{ lb.} - 5 \text{ lb.} - 5 \text{ lb.} - 5 \text{ lb.} = 0$. *Ans.*, 5 lb. This is equivalent to finding the number which used as an addend 3 times will produce 15 lb. In quotition, or measurement, we may find how many times the divisor may be subtracted from the dividend. $15 \text{ lb.} \div 5 \text{ lb.} = ?$ $15 \text{ lb.} - 5 \text{ lb.} = 10 \text{ lb.}$ $10 \text{ lb.} - 5 \text{ lb.} = 5 \text{ lb.}$ $5 \text{ lb.} - 5 \text{ lb.} = 0$. *Ans.*, 3 times. This is equivalent to finding how many times 5 lb. must be used as an addend to produce 15 lb.

¹ Cleveland Survey Test — Set N.

A more advanced method of performing the operation is to find the result through a knowledge of multiplication. In *quotition* we are to find the multiplier. Thus, in $5 \text{ lb.})15 \text{ lb.}$, the question is: 5 lb. multiplied by what equals 15 lb.? *Ans.*, 3. In *partition* we are to find the multiplicand. Thus, in $3)15 \text{ lb.}$, or $\frac{1}{3}$ of 15 lb., we are to find what number multiplied by 3 gives 15 lb.? Or we may say: 3 times what number equals 15 lb.? *Ans.*, 5 lb.

The combinations. — The simplest type of example is that which arises from the combinations in multiplication. There are 90 such combinations: $0 \div 1$, $1 \div 1$, $2 \div 1$, and so on up to $9 \div 1$; $0 \div 2$, $2 \div 2$, $4 \div 2$ up to $18 \div 2$; and so on, up to $81 \div 9$.

To master these combinations so that the results may be called instantly is the foundation work in division. There are two steps necessary: (1) find the results from the corresponding combinations in multiplication, and (2) memorize the results.

Relation to multiplication. — In finding the results care must be taken to select the correct multiplication combination. Thus, in teaching the 6's in *quotition*, $6)6$, $6)12$, $6)18$ up to $6)54$, we need to find how many 6's in 6, 12, 18, and so on; hence we must refer to the multiplication table in which 6 is the constant multiplicand, 1 6 is 6, 2 6's are 12, 3 6's are 18, etc.

A question in *quotition* may be expressed in the following ways: How many 6's in 18? How many times is 6 contained in 18? How many times does 18 contain 6? What is 18 divided by 6 equal to?

In teaching $\frac{1}{6}$ of 6, $\frac{1}{6}$ of 12, up to $\frac{1}{6}$ of 54 (*partition*), we need to find one of the six equal parts of 6, 12, etc.;

hence we must refer to the multiplication table in which 6 is the constant multiplier, 6 1's are 6, 6 2's are 12, and so on up to 6 9's are 54.

Frequent drills are required to memorize these combinations, but since the division facts come directly from the multiplication facts, they need not be studied in consecutive or tabular order. Miscellaneous drill may be had from the first.

Short division. — The next type of example in division has a divisor of one order, but a quotient of two or more orders. This is called short division. The process requires the separation of the dividend into *partial dividends*, each of which may, or may not, be exactly divisible by the divisor.

Full explanations. — In $3 \overline{)963}$, 9 hundreds, 6 tens, and 3 units are each exactly divisible by 3, giving 3 hundreds, 2 tens, and 1 unit respectively. Each one of these smaller examples is *logically* a partition example: *9 hundreds divided by 3 equals 3 hundreds* is analogous to *9 feet divided by 3 equals 3 feet*.

In $3 \overline{)1045}$, the situation is less simple. Here our first partial dividend, 10 hundred, is not exactly divisible by 3. We may proceed in either of two ways:

(a) We may divide 10 hundred by 3, obtaining 3 hundred with 1 hundred remaining. (The fact that our remainder, 1 hundred, at this stage is greater than our divisor need not trouble us as it means that at this point 1 hundred is a part of our dividend that has not yet been divided by 3.) 1 hundred equals 10 tens; 10 tens plus 4 tens then becomes our next partial dividend. 14 tens divided by 3 equals 4 tens with 2 tens remaining; 2 tens

or 20 units plus 5 units equals 25 units ; 25 units divided by 3 equals 8 units with 1 unit remaining. *Ans.* 3h 4t 8u or 348 with 1 remaining. If the last remainder (which must be less than the divisor) is divided by 3, the result is expressed as $\frac{1}{3}$, which gives for the *complete quotient* (see page 107) $348\frac{1}{3}$.

(b) It is perfectly logical to use as partial dividends 9 hundreds, 12 tens, and 25 units, giving us as quotients 3 hundreds, 4 tens, and $8\frac{1}{3}$ units respectively.

Working explanation. — We want the pupil to say as little as possible as he works. It is necessary for him to have each partial dividend in mind ; hence a helpful form is to call off each partial dividend, then the quotient figure for each as he writes it. The pupils' *working explanation* for $3 \overline{)1045}$ by method (a) is 10, 3 ; 14, 4 ; 25, 8 with 1. This procedure requires facility in calling at sight quotients and remainders (see page 125). The *working explanation* by method (b) is 9, 3 ; 12, 4 ; 25, 8 with 1. This requires facility in recognizing, for each partial dividend, the nearest multiple of the divisor which is less than the partial dividend.

Long division. — The most difficult type of division consists in dividing by a number of two or more orders. This is called *long division*. To find the result we proceed, as in short division, to separate the dividend into partial dividends. Thus, in dividing 9882 by 24 we divide 98 hundreds by 24, 28 tens by 24, and 42 units by 24. In short division each quotient figure is found through a knowledge of the combinations, and each remainder is found by performing the multiplication and subtraction mentally. In long division each quotient figure is found

by trial and each remainder is found by performing the multiplication and subtraction as written work.

Illustration :

$$\begin{array}{r}
 411 \\
 24 \overline{)9882} \\
 \underline{96} = 400 \times 24 \\
 28 \\
 \underline{24} = 10 \times 24 \\
 42 \\
 \underline{24} = 1 \times 24 \\
 18
 \end{array}$$

Ans., 411 with 18, or $411 \frac{18}{24}$.

Full, or teacher's explanation. — 98 hundreds divided by 24 equals 4 hundreds and 2 hundreds remaining; we write the 4 in hundreds' place. 2 hundreds, or 20 tens, and 8 tens are 28 tens; 28 tens divided by 24 equals 1 ten and 4 tens remaining, etc. Note that the form of explanation is exactly the same as for short division and necessarily so, since the procedure is the same — the only difference being that in long division every step is performed in written form, while in short division the steps are taken mentally.

Long division may also be regarded as a series of subtractions; first we subtract 9600 or 400×24 ; then 240 or 10×24 ; then 24.

Technique for finding a quotient figure. — Let us consider in more detail the method of finding each quotient figure by trial. In the above example to divide 98 by 24, we may divide 9 tens by 2 tens, $2 \ 4 \overline{)9 \ 8}$, i.e., 9 by 2, obtaining 4. In finding the first quotient figure in $786 \overline{)39864}$ we may divide 39 by 7, obtaining 5. To test this quotient figure before writing it, we multiply 78 by

5, obtaining 390, and compare this result with 398. In this case 5 is the correct first quotient figure. This procedure may be followed for a divisor of any number of figures. Thus in $7946 \overline{)56123569}$, $56 \div 7 = 8$; multiply 79 by 8; this gives 632, which is greater than 561. Try 7; $7 \times 79 = 553$, which is less than 561; hence 7 is the first quotient figure. The *working explanation* is: 56 divided by 7, 8; 8 times 79, 632; 8 is too large; 7×79 , 553; 7 is probably right.

Raising the divisor. — When such divisors as 28 or 29, 18 or 19, etc., are used, many advocate the method of using 30, 20, etc., as the trial divisor. But by this method the quotient figure is sometimes too small. Thus in $591 \overline{)47354}$ if we use 6 as our trial divisor, we find that $47 \div 6 = 7$, whereas 8 is the true quotient figure. It seems to introduce needless confusion to teach two different ways of testing for finding the correct quotient figure, one to determine whether it is too large and another whether it is too small. In mechanizing a process the more uniform our procedure the more efficient our performance. In the above example we say $47 \div 5 = 9$; $9 \times 59 = 531$; try 8; $8 \times 59 = 472$; 8 is right. There is certainly no saving of time in saying $47 \div 6 = 7$; $7 \times 591 = 4137$; $4735 - 4137 = 598$; 7 is too small; try 8. By this method the whole divisor must be multiplied since it is necessary to perform the subtraction before the mistake is disclosed.

The difficulty of introducing a double test does not occur if the uniform procedure advocated above is followed. The only test required is to determine whether the quotient figure is too large; it will never be too small.

The difficulty that arises when several trials are necessary for such trial divisors as 18, 28, 39, etc., may be largely overcome by enough practice in mental multiplication of numbers of two orders by numbers of one order to call products with facility (p. 100). Time thus spent is not wasted, since mental multiplication is needed in finding each quotient figure in any long division example. Thus, in $1872 \overline{)93756}$ instead of saying $9 \div 1 = 9$ and trying in succession 9, 8, 7, 6, 5, we may say at once $93 \div 18 = 5$; $5 \times 187 = 935$; 5 is correct. Had we used 20 and divided 9 by 2 giving 4, we should have had a longer procedure before discovering that 4 is too small, since this fact is not disclosed until the first subtraction has been performed, as stated above.

Division by powers of 10. — Let us consider how to divide a number by 10. Any number of hundreds divided by 10 will give that number of tens, any number of tens divided by 10 will give that number of units, etc., since ten of any order gives one of the next higher order. Hence, *To divide by 10, 100, etc., move each digit one, two, etc., places to the right.* Illustration: $463 \div 10$. The answer may be written as quotient with remainder, 46 with 3; or as a complete quotient, $46\frac{3}{10}$.

Division by multiples of 10^n . — Let us take $76234 \div 400$.

$$\begin{array}{r} 4\ 00 \overline{)762\ 34} \\ 190 \quad \text{with } 234 \end{array}$$

To divide 76234 by 400 is to divide it by 4 and by 100, because $400 = 4 \times 100$. To divide by 100 is to get 762 with 34 remaining, by the rule for dividing by 100. To divide 762 by 4 is to obtain 190 with 2 remaining. The

true remainder is $200 + 34$, because 762 is the number in each of the 100 equal groups; hence the remainder, 2, is the remainder in each of the 100 equal groups; the entire remainder is, therefore, 200 increased by the first remainder, 34, or 234.

Hence the following rule: To divide by a multiple of 10, 100, 1000, etc., first divide by the highest power of 10 which is a factor of the divisor and then divide the result by the other factor of the divisor. To find the remainder multiply the last remainder by the first divisor and add the first remainder.

Short methods. — The work may be abbreviated in long division by performing the multiplication and subtraction together, using the Austrian method for subtracting. This is known as the *Austrian method of long division*. The work is arranged as below:

87	8 times 6 are 48; to 48 enough must be added to
526)46216	give a number ending in 1; 48 and 3 are 51; we
4136	write the 3 and carry the 5 to add in with our next
454	product; 8 times 2 are 16, with 5 to carry are 21;

we must add enough to 21 to give a number ending in 2; 21 and 1 are 22, etc. Our remainder is 413; we then bring down the 6 and continue, as before, to multiply and subtract together.

Working explanation: 48 and 3 are 51; carry 5; 16, 21, and 1 are 22; carry 2; 40, 42, and 4 are 46. Bring down the 6.

To divide by such numbers as 24, 28, 81, etc., we may separate the divisor into factors of one order and divide by the factors in succession, thus resolving the division into a series of short division examples. Thus, to divide by 24 we may divide by 6 and by 4 in succession, or by 8 and by 3. Let us divide 911 by 24.

The true remainder is found as in dividing by multiples of 10ⁿ. 5, the second remainder, is the remainder in each of the 4 equal parts into which 911 was divided. Hence the true remainder is 4 5's + 3 or 23.

$$4 \overline{)911}$$

$$6 \overline{)227}, 3$$

$$37, 5$$

Ans. 37 with 23.

Mental division. — Proceed as in ordinary short division, calling off each partial dividend and the corresponding quotient figure. Illustration: In $497 \div 6$, call off 49, 8; 17, 2 with 5. Ans. $82\frac{5}{6}$ or 82 with 5.

Methods of checking. — The common check is to multiply the quotient by the divisor and add the remainder; the result equals the dividend. This is true because if the remainder be subtracted from the dividend, the result is the product of the divisor and quotient, by definition of division.

Since the product of the divisor and quotient plus the remainder equals the dividend, to prove by 9's, we apply the check by 9's for a multiplication example, and then for an addition example. Hence, the rule:

To check division by 9's, multiply the excess of 9's in the quotient by the excess of 9's of the divisor; add the excess of 9's of the remainder; the excess of 9's in this result should equal the excess of 9's in the dividend. The work is arranged as follows:

61	7	The excess of 9's in 61 is 7; in 782 is 8;
782 $\overline{)48151}$	8	the product of 8 and 7 is 56. The excess of
4692	$\overline{)56}$	9's in 449 is 8; $56 + 8 = 64$; the excess in
$\overline{)1231}$	8	64 is 1; this is the same as the excess in the
782	$\overline{)1}$	dividend.
$\overline{)449}$		

Precedence of signs. — In an expression containing the four fundamental operations, it has been agreed that

the operations of multiplication and division shall take precedence over addition and subtraction. Thus, in such an expression as $4 + 5 \times 3$, the multiplication is performed first. The value of the expression is, therefore, $4 + 15$ or 19. In $16 - 8 \div 4 + 3 \times 2$, we first find the value of $8 \div 4$ and of 3×2 ; the expression equals $16 - 2 + 6$ or 20. If the addition or subtraction is to be performed first, parentheses must be employed to indicate that fact. Thus, $(16 - 8) \div (4 + 2)$ would equal $8 \div 6$ or $\frac{4}{3}$.

Mechanical computation. — An instrument which automatically produces the results of number combinations involving the union of different orders is called a calculating machine. Modern calculating machines are divided into two large classes, adding machines, and multiplying and dividing machines.

One form of the adding machine is called the *comptometer*. Numbers are added by depressing keys similar to those on a typewriter. The mechanism consists in placing the units around the circumference of a wheel, the tens in same manner on a second wheel, the hundreds on a third wheel, etc. The wheels are operated by keys which cause them to turn one-tenth, two-tenths, etc., corresponding to the key which has been depressed. At the end of a complete revolution, the unit wheel engages the tens' wheel, causing it to move up to the one-tenth division; in the same way the tens' wheel is connected with the hundreds', etc.

Subtraction is performed by adding the complement of the number. Multiplication is performed by repeated additions. Thus, to multiply 56 by 24, strike 56, 4 times; then 560, 2 times. In these machines the results appear

at the front, but the numbers which make up the result are not shown. Some types of machines print the items as well as the results; these are known as listing machines. The Burroughs adding machine is of this type.

Multiplying machines in which the number to be multiplied is set up, and a small handle then turned as many times as the multiplier indicates, were conceived of by Leibnitz as early as 1671, but not perfected sufficiently to be practical until 1820. A multiplying machine which performs the multiplication directly, applying the multiplication table, was constructed by Bollée in 1889. A more modern machine of this type is called the Millionaire, invented in 1892. To multiply 598 by 36 by this machine 598 is set by sliding small knobs, then a pointer is turned to 3, an electric button is pressed; the pointer is turned to 6, the button pressed again and the product appears on a special row of dials. Division is performed by reversing the mechanism.¹

HISTORICAL NOTES

Earliest methods.² — Among the ancients division was regarded as an exceedingly difficult process. As with multiplication, so with division, only those nations that attained a high degree of civilization employed the process at all. Division tables, as well as multiplication tables, appeared on the Nippur tablets.

Al-Khowârizmî, an Arabic writer who lived about

¹ LOCKE, L. LELAND. — *The Mathematics Teacher*, XV, No. 7; *The American Mathematical Monthly*, XXXI, No. 9.

² BALL, W. W. — *Short History of Mathematics*, pp. 197-201. CAJORI, F. — *History of Elementary Mathematics*, pp. 98, 105; *History of Mathematics Revised and Enlarged*, pp. 7, 117. SMITH, D. E. — *History of Mathematics*, Vol. I, pp. 40, 255.

800 A.D., used a method of division in which the divisor was written at the bottom, the quotient above the dividend, and, above the quotient, the successive results of subtracting partial quotients. The illustration is from Cajori¹ and shows the division of 46468 by 324.

The Greeks and Romans performed division by subtraction, using the abacus to record the results of the successive subtractions of the divisor or a multiple of the divisor.

Italian methods. — Two methods of long division were in use among the early Italians : the so-called *galley method* and our present method, called *division a danda* (from something given). Pacioli, in his arithmetic, published in 1494, gives these two methods and two methods for short division. One method of short division is practically the same as that used to-day ; the second is division by successive factors of the divisor.

The first printed example in long division by our present method was in Calandri's arithmetic published in 1491.² Gerbert, who was a noted mathematician of the latter part of the tenth century, gave a rule for division so difficult that but few could understand it. The galley method was in use when work was done on sand tablets and figures were scratched out as soon as used. After printing came into use this method died out, although it was used for several centuries after our present method was well known. The following example illustrates the successive steps in dividing 59078 by 74 by the scratch or galley method.

¹ *History of Elementary Mathematics*, p. 105.

² SMITH, D. E. — *History of Mathematics*, p. 255.

$\begin{array}{r} 59078 \overline{)74} \\ 74 \end{array}$	$\begin{array}{r} 10 \\ 59078 \overline{)74} \end{array}$
$\begin{array}{r} 7 \\ 102 \\ 59078 \overline{)744} \\ 744 \\ 7 \end{array}$	$\begin{array}{r} 6 \\ 79 \\ 1021 \\ 59078 \overline{)79444} \\ 7444 \\ 77 \end{array}$
	$\begin{array}{r} 62 \\ 793 \\ 10216 \\ 59078 \overline{)79844} \\ 7444 \\ 77 \end{array}$

EXERCISES

1. The cost of three two-cent stamps is 6¢. Write the corresponding quotient problem and solve it (a) by counting objects, (b) by addition, (c) by subtraction.

2. In playing whist the 52 cards are dealt one by one to the 4 players. How many cards does each player receive? Which aspect of division does this illustrate?

3. Write the 90 combinations in miscellaneous order. Call off the results, recording your time. Then write the results, recording your time. See Exercise 1, page 36.

4. Averaging 24 miles per hour, how far will an automobile travel in 4 hours? State the related problem (a) in quotient, (b) in partition.

5. Formulate a problem which involves (a) quotient with no remainder, (b) quotient with a remainder, (c) partition with a remainder, (d) partition with no remainder, quotient a mixed number.

6. From the addition here shown what multiplication can be derived? From this multiplication fact, what division facts can be derived? Which of these is partition and which quotient (measurement)?

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7. Tell what each means and give the result: 21 bu. $\div$ 3; \$18 $\div$ \$6; 13 ft. $\div$ 3; 19¢ $\div$ 4¢.

8. Show devices for oral drill in preparation for short division by 7. Tell how you would use them.

9. Show by using sticks the process of dividing 748 by 3. HINT. (7 h 4 t 8 u) $\div$ 3.

10. In $7\overline{)361}$, what is the remainder? The complete quotient? The integral quotient?

11. Show that the steps in long division are the same as those in short division. Use the examples $3\overline{)201}$ and $31\overline{)2077}$.

12. For each of the following examples give (a) the complete explanation, (b) the working explanation: $3\overline{)2469}$, $3\overline{)2016}$, $3\overline{)17426}$, $3\overline{)2724}$.

13. Divide 5784326 by 8623, obtaining quotient and remainder. Check by 9's. To the right of each subtrahend write the number of times the divisor which the subtrahend equals.

14. Check the division in example 13 by the common method, using the divisor as the multiplicand. Compare the partial products in the check with the subtrahends in the division process.

15. For each of the following examples (a) find the quotient and remainder, (b) write the exact quotient, (c) give the working explanation for finding the quotient figures.

(1) $15485 \div 38$

(3) $578482 \div 8634$

(2) $2841905 \div 473$

(4) $525472806296 \div 768294$.

16. Using the method on page 114, find the quotient and remainder for each of the following: $2374 \div 40$, $68324 \div 700$, $28973 \div 360$. Explain in full.

17. Work the examples in Exercise 15 by the Austrian method.

18. Work the following examples by means of two short divisions: $7428 \div 56$; $1237 \div 18$; $5284 \div 48$.

Express each result (a) in the form of a quotient with remainder, (b) as an exact quotient.

19. Call off the results. Do not write them.

$56 \div 8$

$41 \div 9$

$117 \div 9$

$52 \div 4$

$72 \div 8$

$59 \div 7$

$136 \div 8$

$64 \div 4$

$36 \div 9$

$51 \div 8$

$98 \div 7$

$105 \div 7$

$56 \div 7$

$98 \div 11$

$84 \div 6$

$70 \div 5$

$45 \div 5$

$78 \div 9$

$96 \div 4$

$72 \div 4$

$54 \div 9$

$86 \div 7$

$72 \div 3$

$65 \div 5$

$110 \div 11$

$47 \div 6$

$91 \div 7$

$56 \div 4$

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20. Write the results for the previous exercise, recording time.

21. Find the average amount of sugar produced per year for the 7 years (a) from cane, (b) from beets, (c) total.

SUGAR

YEAR	CANE SUGAR	BEETS, POUNDS	TOTAL
1917	489,437,760	1,530,414,000	
1918	571,056,640	1,521,900,000	
1919	241,998,400	1,452,902,000	
1920	352,204,160	2,178,000,000	
1921	655,399,360		2,696,377,360
1922	628,630,726		1,978,630,720
1923	472,000,000		2,176,000,000
Total			
Av. per yr.			

22. On September 3, 1924, Lieut. Moffett of the Boston air post flew to Mitchell Field, New York City, and returned in 2 hours 5 minutes of actual flying time. The total distance is 366 miles. Find the rate per hour; the rate per minute.

METHODS OF TEACHING

The combinations. — On page 79 there are shown two organizations of the multiplication combinations: a table having a constant *multiplicand* and a table having a constant *multiplier*. The second organization serves as the basis for the process of written multiplication; the first organization as the basis for the process of short division.

From the fact that $2 + 2 + 2 = 6$, it follows that 3 2's, or 3 times 2, are 6; also that 6 contains 2 three times, and $\frac{1}{3}$ of 6 equals 2. Note that in the first division combination the quotient, 2, is one of the three equal parts of

the dividend. While it is entirely unnecessary, if not impossible, to get children to appreciate the difference between these two aspects of division, it is perfectly easy for the teacher to understand these relations and to observe them in her teaching. The children meet these relations in their problem work and are able to sense the difference in question because there is a concrete situation in the problem, while the combination is abstract.

It is easily seen that the multiplication table which is derived from the columns in which 2 is the constant addend will yield a division table in which 2 is the constant divisor, and that any quotient thus obtained expresses a *number of times contained*. It is this set of combinations which is used in short division by 2. In teaching, it is important to familiarize the pupils with the various ways of expressing the relations. To illustrate with one case:

3 times 2 are 6, or 3 2's are 6.

There are 3 2's in 6.

6 contains 3 2's.

6 contains 2, 3 times.

2 is contained in 6, 3 times.

$6 \div 2 = 3$.

$$\begin{array}{r} 2 \overline{)6} \\ 3 \end{array}$$

Let us illustrate a good procedure for developing such a table; we shall take the 3's.

1. Give a drill exercise on the multiplication combinations of *times 3*; the children say: Four 3's (or 4 times 3) are 12, etc. Finally the products are written outside the circle. (Figs. 15 and 16.)

2. The device now looks like Fig. 16. Ask the children to cover

their eyes; erase one of the figures within the circle and ask: What have I erased? Some child answers: 2. The teacher asks: How



FIGURE 15

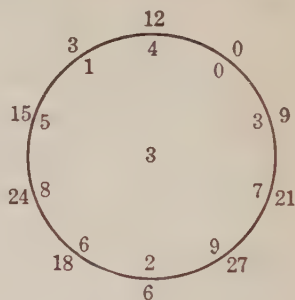


FIGURE 16

do you know? Answer: Because 2 times 3 are 6. Proceed in like manner with the other combinations.

3. The device now looks like Fig. 17. Now take up drill on the combinations, using in succession the first five forms of expression illustrated on p. 123 in the case of 3 times 2 are 6.

4. Next, write the combinations in the forms shown below and in miscellaneous order. Have the children read the combinations thus:

Twelve divided by three equals four, etc.

$$12 \div 3 = \quad 18 \div 3 = \quad 9 \div 3 =$$

$$6 \div 3 = \quad 27 \div 3 = \quad 15 \div 3 =$$

$$0 \div 3 = \quad 3 \div 3 = \quad 24 \div 3 =$$

$$21 \div 3 =$$

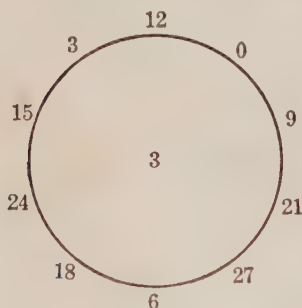


FIGURE 17

5. Finally, write the combinations in the form shown on the next page, and in miscellaneous order. Have the children at first read the combinations as they were read in step 4, saying: *Eighteen divided by three equals six, etc.*; when these responses come without hesitancy, shorten the time by having the children call the quotients only.

$$\begin{array}{cccccc}
 3 \overline{)18} & 3 \overline{)15} & 3 \overline{)24} & 3 \overline{)9} & 3 \overline{)6} & 3 \overline{)0} \\
 & 3 \overline{)12} & 3 \overline{)3} & 3 \overline{)21} & 3 \overline{)27} &
 \end{array}$$

By far the major portion of the drill on the combinations should make use of the short division form, the arrangement in step 5.

The development of the partition combinations starts with the multiplication table in which the multiplier is constant. The essential steps are illustrated thus: $4 + 4 + 4 = 12$, one of the three equal parts of 12 is 4, $\frac{1}{3}$ of 12 = 4.

Sight division with remainders. — As soon as the children know the combinations of the *threes* they can apply them directly in working examples like $3 \overline{)69}$, $3 \overline{)183}$, $3 \overline{)276}$, in which each partial dividend is an exact multiple of the divisor; but before other examples can be worked there must be instruction and drill on calling at sight quotients and remainders for dividends which are not multiples of the divisor. The device here shown displays to the children just what these combinations are, and furnishes a convenient means for carrying on the early instruction and drill.

0	1	2
3	4	5
6	7	8
9	10	11
12	13	14
15	16	17
18	19	20
21	22	23
24	25	26
27	28	29

While there is no fixed procedure, the following is one way of using this arrangement:

The children give, in regular, then in miscellaneous, order the quotients of the first column; next, they give the quotients and remainder of the second column first in regular, then in miscellaneous, order; similarly for the remaining columns. They now understand that in any given column the remainder is constant — in other words, that every number is just so much more than some multiple of the divisor.

The children now proceed by lines, calling the results first in regular, then in miscellaneous, order. This emphasizes the fact that in each line the quotient is constant, and the remainders vary from 0 to 1 *less than the divisor*. Finally, the children are expected to call the results as the teacher points at random to any numbers in the field. Miscellaneous practice without the chart should follow.

Short division. — Here the children are faced with two new difficulties: (1) that of separating the dividend into partial dividends, each of which has appeared in the above drill in sight division, and (2) that of coördinating the proper motor responses of writing the quotient figures with the calling of the quotients and remainders. Here, as with the early work in written multiplication, the success of the pupils depends to a very great degree upon the right habituation of the procedure. The character of the finished complex habit (working explanation) is discussed and illustrated on page 111, but it would be futile to expect children to use this habit at the outset. The procedure involves (1) thinking the partial dividend, (2) thinking quotient and remainder and writing the quotient. Each partial dividend, except the first, must be formed by visualizing the remainder before the next figure of the dividend. At first it is necessary for the child to give conscious attention to every element of the process, and this requires painstaking drill in which the children are required to think through, and express fully in words, the steps of the process. As soon as any element of the procedure comes to be handled with confidence, the oral expression of that element should be omitted; this means that it is in the way of being taken care of subconsciously.

Let us illustrate the progressive steps which might be taken in mechanizing the process in connection with the following examples:

$$4 \overline{)136}, 4 \overline{)184}, 4 \overline{)232}, 4 \overline{)262}, 4 \overline{)116}.$$

1. Have the children call the partial dividends, quotients, and remainders — the teacher writing the quotient figure as it is called. Thus: 13, 3 with 1; 16, 4; etc.

2. Same as in 1, except that the children write the quotient figures.

3. Have the children call the partial dividends and quotients, the teacher writing each quotient figure as it is called.

4. Same as in 3, except that the children write each quotient figure as it is called.

5. Have the children call the partial dividends and do all the written work.

6. Have the children work examples silently, but under careful supervision, until the mechanical habit is well established.

It would seem that the thinking of the partial dividends must always be a conscious step, and the other steps should, with the more capable pupils, be taken subconsciously. When the children have a fair command of the process, examples involving a final remainder may be given; these require very little teaching — merely tell the pupils to write the remainder thus:
$$\begin{array}{r} 4 \overline{)379} \\ 94 \text{ with } 3. \end{array}$$

Dividends of four figures involve nothing new; they are deferred for a time until the mechanics are well in hand.

Common check. — The children should be taught to prove, or check, division by multiplication, adding the remainder if any. It is doubtful if the children can be made to feel the necessity for accurate work for its own sake, but as they grow older the motive of pleasing their teacher can be displaced by degrees and the real meaning of checks appreciated.

Long division explained. — Long division is generally considered the special topic for the fourth year. Few teachers make any attempt to rationalize this process. The common check, already familiar through its use in connection with short division, will furnish sufficient evidence that the long division process is correct ; and here again the use of problems, by giving a better sense of the reality of the relations, will serve to strengthen this evidence. To illustrate : A school has 1008 pupils ; there are 42 children in a room. How many rooms in the school ? The process of long division gives the result 24 rooms. Now the child checks this result by multiplying to see if 24 rooms having 42 children in each room will give 1008 children. His discovery that this is the fact gives him a confidence in the process which the same work, without the problems, could not give. It is possible in a later grade to study the *nature* of the process of long division as follows. Use the problem : If a train travels 43 miles per hour, in how many hours will it travel 2021 miles ?

SOLUTION OF ORIGINAL PROBLEM

$$\begin{array}{r}
 47 \\
 43 \overline{)2021} = \text{total number of miles.} \\
 \underline{172} = 40 \times 43, \text{ or number of miles in 40 hours.} \\
 301 = \text{number of miles yet to go.} \\
 \underline{301} = 7 \times 43, \text{ or number of miles in 7 hours.}
 \end{array}$$

SOLUTION OF CHECK PROBLEM

$$\begin{array}{r}
 43 = \text{number of miles in one hour.} \\
 47 \\
 \underline{301} = 7 \times 43, \text{ or number of miles in 7 hours.} \\
 \underline{172} = 40 \times 43, \text{ or number of miles in 40 hours.} \\
 2021 = 47 \times 43, \text{ or number of miles in 47 hours.}
 \end{array}$$

The process of long division is here seen to be an abbreviation of the subtraction method (see page 108) of finding a quotient. Instead of subtracting 43 miles from 2021 miles, then 43 miles from the remainder, and so on for 47 times, we have performed only two subtractions; the subtraction of *40 times 43* takes the place of the subtraction of *43 forty separate times*, and the subtraction of *7 times 43* takes the place of the subtraction of *43 seven separate times*. The process of multiplication, used in solving the check problem, involves the inverse steps; instead of using 47 addends, each 43, two addends suffice; the addition of *7 times 43* takes the place of the addition of *43 seven separate times*, and the addition of *40 times 43* takes the place of the addition of *43 forty separate times*.

Many teachers use the following scheme as a partial rationalization of long division.

First work an example in short division by the ordinary process, next work the same example, using the long division form and pointing out that the steps are the same except that they are more fully written, and finally work a long division example in which much the same combinations occur. In this development the mechanics of the two processes are shown to be in all essentials identical.

$\begin{array}{r} 3 \overline{)134} \\ 44 \text{ with } 2 \\ \hline \text{or} \\ 44 \text{ with } 2 \\ 3 \overline{)134} \end{array}$	$\begin{array}{r} 44 \text{ with } 2 \\ 3 \overline{)134} \\ 12 \\ \hline 14 \\ 12 \\ \hline 2 \end{array}$	$\begin{array}{r} 42 \\ 32 \overline{)1344} \\ 128 \\ \hline 64 \\ 64 \\ \hline \end{array}$
---	---	--

While the children are struggling to master the steps it is well to have them written on the blackboard in plain view, thus:

1. Get the partial dividend.
2. Find the quotient figure, and write it.

3. Multiply the divisor by the quotient figure.
4. Subtract this product from the partial dividend.
5. Compare the remainder with the divisor ; the remainder must be less than the divisor.

Or the steps may be indicated in catch-word form.

1. Partial dividend.
2. Quotient figure.
3. Multiply.
4. Subtract.
5. Compare.

When divisors of three or more orders are taken up, the process of finding the quotient figure (see page 112) requires careful teaching. Full explanation by the teacher should be followed by daily drill on the working explanation. The children should not be permitted to work unsupervised until the method is well established as a habit. Some teachers suggest shortening the work of finding the quotient figure when the next to the highest order of the divisor is occupied by 8 or 9. To illustrate: In the example shown, to follow the rule on page 112 would involve four trials, 9, 8, 7, 6, before the correct quotient figure would appear ; but divid- $293 \overline{)1984}$ ing 19 by 3 (considering the divisor as about 300) would give the correct quotient figure at once. However, the wise teacher will avoid urging this method upon the children. The disadvantages of this procedure are discussed on page 113. In the higher grades the more capable children are likely to depart more or less from the regular rule, not only in the manner just described but by dividing by the number expressed in the two highest orders of the divisor. Any individual departure from the

regular procedure, if it improves the child's technique, should be commended by the teacher, but it would be a waste of time for the teacher to urge such departures upon his pupils.

Types of examples in long division. — In listing the types of examples with reference to degree of difficulty it is necessary to consider first the elements which give rise to differences in difficulty. These elements are :

1. Number of figures in the divisor. There may be 2, 3, . . . figures.

2. Number of figures in the quotient. There may be 1, 2, . . . figures.

3. Number of trials to find the quotient figures (see page 112).

4. Presence of 0's in the quotient — in units' order, in some higher order or orders.

Some teachers distinguish, and give conscious attention to, upwards of twenty degrees of difficulty in long division ; at the other extreme are a few who at the outset teach children to work examples with three-order divisors, two- or three-order quotients, and involving trouble in finding quotient figures. The best practice is to avoid these extremes. The teacher should make sure that all the difficulties occur both singly and in combination in the examples presented. Formal instruction on the main types and on other types should be given when the children have become conscious of a difficulty.

Comptometers and their influence upon arithmetic teaching. — In banks, in the offices of large commercial and industrial concerns, and wherever there is much computing to be done, we find the work being done by ma-

chinery. The machine most commonly used is the adding machine. While all of the fundamental operations can be performed upon the adding machine, it is a great saving of time to use a calculating machine for multiplying (and dividing also) where there is much of this to be done. Some educators have expressed extreme views as to the changes that should be made in the teaching of the operations because of the increasing use of comptometers. It has been asserted that there is little use of children learning to compute with accuracy and speed by the traditional method; it would be better to omit entirely the greater part of the usual mechanical drill, or else teach children the use of comptometers. The facts to be considered in this connection are these: (1) As yet only a comparatively few people have any occasion to use comptometers. (2) Everybody has more or less computing to do, and in most instances accuracy, rather than high speed, is indispensable. (3) Accuracy and a sense for numerical relations are required alike in the operator of a calculating machine and in the ordinary mortal. (4) The great majority of computations that occur in the experience of the general public have to do with small numbers.

Wilson's study¹ shows that of 1737 additions occurring in common experience 1050 involved only two or three addends, and only 26 involved more than ten addends; of 2833 subtractions 1152 involved one- or two-figure minuends, 1101 involved three-figure minuends; of 6974 multiplications 3705 involved one-figure multipliers,

¹ WILSON, G. M. — "A Survey of the Social and Business Usages of Arithmetic"; *Teachers College Contributions to Education*, No. 100, 1919.

2733 involved two-figure multipliers ; and of 2437 divisions 966 involved one-figure divisors, 1058 involved two-figure divisors. But it also appears that people occasionally have to deal with very long examples — with additions of twenty or more addends, with subtractions in six orders, with multiplications and divisions by numbers of four orders.

In view of these facts it is plainly the business of arithmetic teachers to develop real skill in mental computations, to inculcate habits of accuracy and the power of sustained attention in longer examples, the habit of applying appropriate checks, and common sense about numbers. There are resting places in long multiplication and division examples ; subtraction examples are relatively short at their worst, and long addition columns can be broken up into short examples, thus avoiding the strain most pupils feel in adding very long columns. Speed should not be carried to the point where it results in a painful sense of hurry and anxiety. The sprinter is not developed by practicing at top speed, but he is carefully trained with a view to developing confidence in his ability to run a good race ; good technique and high morale are the chief elements in the process.

EXERCISES

1. Write the first four combinations in the table of 6 times ; write the corresponding *quotition* combinations, also the corresponding *partition* combinations.
2. Write the first four combinations in the table of times 6 ; write the corresponding *quotition* combinations, also the *partition* combinations.
3. In which of the above exercises does a serviceable partition

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table appear? A serviceable quotation table? In what respect do these two tables selected as serviceable agree?

4. Make a detailed plan for teaching the *sizes* in quotation. Include instruction, drill, and application.

5. Make a detailed plan for teaching sight division by 6 with remainders.

6. Make a detailed plan for teaching short division by 6.

7. Using a pair of related problems, plan the first lesson on checking division by multiplication.

8. Plan a preparatory oral drill to precede drill work in short division by 6.

9. Following the suggestions on page 131, prepare a series of examples illustrating all the degrees of difficulty in long division. For each example state what difficulty or combination of difficulties is present.

10. Take some textbook in which the first term's work in long division is presented. Note (a) the *order* in which the various difficulties are presented, (b) the number of examples affording practice on each difficulty. If you were to use this textbook, would you have your class work the examples in the textbook order?

11. Discuss the rationalization of long division, using the following questions: Is it possible? If possible, is it desirable? If possible and desirable, what is the best method? Draw upon your knowledge of psychology in defending your position on this matter.

12. Plan in detail the first lesson on long division, the aim being to lead up to the five general steps in the mechanical process.

13. Prepare an example having a three-figure divisor which involves two trials to find the first quotient figure; also a similar example requiring three trials. Explain as to a class the process of finding a quotient figure.

14. Prepare, or select from some book, a list of ten examples for drill in finding quotient figures according to the method described on page 112. Tell how you would proceed in conducting this drill.

15. Prepare a list of ten examples for drill on any special methods for finding quotient figures in long division.

16. Describe any special method you have found helpful in con-

nection with finding quotient figures. Discuss its value. Would you teach it?

17. Prepare three long-division examples which give three-order quotients with 0 in tens' order. How would you teach this difficulty?

18. Prepare three examples whose quotients have 0 in units' place. How would you teach this difficulty?

19. Which of the last two exercises would you teach first? Why?

CHAPTER VI

FACTORS AND MULTIPLES

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions. — A number may be expressed as the exact product of two or more integers; these integers are called the *factors* (*facere*, to make) of the number. Any one of the integers which are multiplied together must be an exact divisor of the product. For instance, 3 and 4 are the factors of 12, because $3 \times 4 = 12$; also, each number is exactly contained in 12. Hence:

A *factor* of a number is an exact divisor of it. Or, a factor of a number is one of two or more integers which multiplied together produce the number. Although a factor of a number is always an exact divisor of it, the terms *factors of a number* and *exact divisors of a number* are not synonymous. The expression *the factors of a number* means those numbers which multiplied together produce the original number; the *exact divisors of a number* include all the numbers which exactly divide the given number. To illustrate, the factors of 30 are 5 and 6, or 3 and 10, or 2 and 15, or 2, 3, and 5. The exact divisors of 30 are 2, 3, 5, 6, 10, and 15.

A factor may exactly divide two or more numbers. Such is called a *common factor* of the numbers. Thus, 84 and 28 have the common factors 2, 4, 7, 14, and 28.

The *highest common factor* (H.C.F.) of two or more numbers is the highest number which is an exact divisor of each of them; *e.g.*, 28 is the H.C.F. of 28 and 84.

Numbers may be classified as to their resolvability into factors. Some numbers have no other factors but themselves and 1, as 13 which equals 13×1 ; some have other factors besides themselves and 1, as 14 which equals 2×7 . The first are called *prime* numbers (*primus*, first or original), the second are called *composite* numbers (composed of factors).

The product of two or more integers is called a *multiple* of each of them. Since this product exactly contains each of the integers, we may define a multiple thus:

A *multiple* of a number is an exact container of it. Or, a multiple of a number is the result of multiplying it by any integer including 0. *E.g.*, 0, 4, 8, 12, etc. are the multiples of 4. In the natural series of numbers (1, 2, 3, 4, . . .) every second number is a multiple of 2, every third number is a multiple of 3, every *n*th number is a multiple of *n*.

A *common multiple* of two or more numbers is a number that is a multiple of each of them. For example: 12, 24, 36, etc. are common multiples of 4 and 6; 36, 72, etc. are common multiples of 4, 6, and 9.

The *lowest common multiple* (L.C.M.) of two or more numbers is the lowest number that is a multiple of each of them. 18 is the L.C.M. of 6 and 9. It is often convenient to be able to tell whether a large number is a multiple of smaller numbers, such as 2, 3, 4, 5, etc. without actually performing the division. Rules for this purpose may be discovered as in the following sections.

Divisibility by 2. — *Induction.* — We wish to discover a rule for exact divisibility by 2. Let us obtain some numbers which are exactly divisible by 2 by taking any numbers at random and multiplying by 2. Multiplying 35, 273, and 4637 by 2, we obtain 70, 546, and 9274. Examining these numbers we find the last digit in each (0, 6, 4), is exactly divisible by 2. Hence the probable rule: A number is a multiple of 2 if its units' digit is a multiple of 2.

Deductive proof. — Let us take one of these numbers and study it.

$$9274 = 927 \times 10 + 4$$

Any number may be separated into a number of 10's and a number of units.

927×10 is a multiple of 2.

Because 10 is a multiple of 2.

Therefore 9274 is a multiple of 2.

If each of two numbers is a multiple of 2 their sum must be a multiple of 2.

Divisibility by 4. — *Induction.* — We wish to discover a rule for exact divisibility by 4. Let us obtain some numbers which are exactly divisible by 4 by taking numbers at random and multiplying them by 4: $4 \times 37 = 148$, $4 \times 243 = 972$, $4 \times 3697 = 14788$, etc. Note the numbers denoted by the two right-hand digits: 48, 72, 88; each one is exactly divisible by 4. Hence the probable rule: A number is a multiple of 4 if the last two digits denote a multiple of 4.

For the *deductive* proof see Exercise 27, page 153.

Divisibility by 8. — *Induction.* — Proceeding as with 2 and 4, we discover the probable rule: A number is a multiple of 8 if the number denoted by the three right-hand digits is a multiple of 8.

For *deductive* proof see Exercise 28, page 153.

Divisibility by powers of 5. — *Induction.* — A number is a multiple of 5 if the right-hand digit is exactly divisible by 5. The proof is the same as for the rule for divisibility by 2.

A number is a multiple of 25 if the number denoted by the two right-hand digits is exactly divisible by 25. The proof is the same as for the rule for divisibility by 4.

The rule for divisibility by powers of 2 or 5 may be summarized in the following general rule :

A number is a multiple of 2^n , or of 5^n , if the n right-hand digits denote a multiple of 2^n , or of 5^n , respectively. This rule is true because the other part of the number is a multiple of $10^n (M \times 10^n)$, and $M \times 10^n = M \times 2^n \times 5^n$.

Divisibility by 3. — *Induction.* — If we proceed as in the preceding cases, using numbers which are exactly divisible by 3, such as 9276, 375, etc., we discover that $9 + 2 + 7 + 6$ or 24 is divisible by 3; and that $3 + 7 + 5$ or 15 is divisible by 3. Hence the following probable rule : A number is a multiple of 3 if the sum of its digits is a multiple of 3.

In the deductive proof we may use a single number¹ or

¹ Let us take any number the sum of whose digits is exactly divisible by 3; *e.g.*, 432.

$$432 = 4 \times 100 + 3 \times 10 + 2$$

$$= 4 \times (99 + 1) + 3$$

$$\times (9 + 1) + 2$$

$$= 4 \times 99 + 4 + 3 \times 9 + 3 + 2$$

$$= (4 \times 99 + 3 \times 9)$$

$$+ (4 + 3 + 2)$$

$(4 \times 99 + 3 \times 9)$ is exactly divisible by 3.

$(4 + 3 + 2)$ is a multiple of 3.

Therefore the whole number, 432, is a multiple of 3.

Any number may be separated into its units, tens, hundreds, etc.

Any power of 10 is a multiple of 3, plus 1.

Removing parentheses, by multiplying each number by 4 in the first and by 3 in the second.

Rearranging the terms, and grouping the multiples of 3.

The sum of two multiples of a number is a multiple of the number.

The number was so taken.

The sum of the multiples of a number is a multiple of the number.

we may use a general expression¹ for all integers. In each method the proof is general because the reasons upon which the statements are based are general.

Divisibility by 9. — *Induction.* — Proceeding as in the previous proofs for 3, we discover the following rule: A number is exactly divisible by 9 if the sum of its digits is exactly divisible by 9. Thus: 728,964 is exactly divisible by 9 because $7 + 2 + 8 + 9 + 6 + 4$, or 36, is exactly divisible by 9.

Divisibility by 11. — *Induction.* — As in other cases take some numbers of four or more orders which are divisible by 11 and develop the following rule:

A number is a multiple of 11 if the difference between the sum of the digits in the odd orders and the sum of the digits in the even orders is a multiple of 11. Thus 2684 is a multiple of 11, because $(2 + 8) - (6 + 4) = 0$, a multiple of 11.

For the *deductive* proof see Exercise 21, page 152.

Divisibility by numbers higher than 11. — If a number is exactly divisible by both 3 and 4, it is evidently divisible by their product, 12; if divisible by 3, 5, and 7, it is evidently divisible by their product, 105. But a number divisible by both 2 and 4 is not necessarily divisible by 8. Hence, if each of two or more numbers *which have no common factor* is a factor of a given number, then the product of these numbers must be a factor of the given number.

¹ Let u be the units' digit, t the tens' digit, h the hundreds' digit, T the thousands' digit, etc., in any integer. Then, any integer may be expressed as follows:

$$u + 10t + 100h + 1000T \text{ and so on} = u + 9t + t + 99h + h + 999T + T \text{ and so on} = (9t + 99h + 999T + \dots) + (u + t + h + T + \dots).$$

Since the quantity $9t + 99h + 999T$ is exactly divisible by 3, the divisibility of the whole number by 3 will depend upon the divisibility of the quantity $u + t + h + T$ which represents the sum of the digits. Hence the rule.

Finding prime numbers. — More than 2000 years ago a Greek, Eratosthenes by name, invented what is known as the “sieve” method of separating prime numbers from composites. This method, aside from its historic interest, is of value in that out of it grows a very neat method of determining whether a number is prime.

To use the “sieve,” write in order the integers from 1 to any desired number, arranging in lines and columns. Cross out every second number after 2 (4, 6, 8, . . .), cross out every third number after 3 (6, 9, 12, 15, . . .), then each

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	etc.			

fifth number after 5 (5, 10, 15, 20, 25, 30, 35 . . .); and each seventh number after 7. It should be noted that the multiples of 3 which are also multiples of 2 (6, 12, 18, etc. . . .) are already crossed out, and that *the first number to be crossed is the square of 3*. Similarly, the first number to disappear that has not already been crossed out when the multiples of 5 are considered is the square of 5. In general, by the sieve method, the first multiple of any prime to be crossed out will be the square of that prime, because the lower multiples of the number will have disappeared in crossing out the multiples of 2, 3, 5, and so on up to the number being considered.

To determine by the sieve method whether any number such as 277 is a prime, it would be necessary to write all the numbers in succession as far as 277; then cross out

the multiples of 2, 3, 5, 7, 11, and 13, respectively. There would be no multiple of 17 remaining to be crossed out, because its first uncrossed multiple would be the square of 17 or 289, a larger number than 277. If 277 remained uncrossed, it would be a prime.

Let us now illustrate the process of determining whether a number is prime by a shorter method. Take the number 277. We use in succession as trial divisors 2, 3, 5, 7, 11, 13, 17. The rules for divisibility are employed in the testing for 2, 3, 5, and 11, and actual division for the other numbers. With 17 as a divisor the quotient is less than the divisor. The number, 277, must be prime, for if there be any factor greater than 17, there must be a factor (quotient) less than 17. But we have tried all the primes less than 17, and the composites which are multiples of these primes certainly could not be contained. Hence 277 must be a prime. To determine whether a number is a prime try as divisors all the prime numbers in succession, beginning with 2, until that prime has been tried whose square is greater than the number being tested. If no exact divisor is found, the number is a prime.

Factoring a number. — By factoring a number is commonly meant the resolving of the number into its *prime* factors. In the last section it was discovered that, if the multiples of all lesser primes have been cancelled, the first multiple to be cancelled with any prime is the square of that prime. This principle is the logical basis of the following rule:

To factor a number, try as divisors the successive prime numbers beginning with 2 until an exact divisor is found,

$$\begin{array}{r}
 16 \\
 17 \overline{)277} \\
 \underline{17} \\
 107 \\
 \underline{102} \\
 5
 \end{array}$$

or until an inexact divisor yields a quotient less than the divisor. If no exact divisor is found, the number is prime. If an exact divisor is found, treat the resulting quotient in the same way, *except* that the first prime to be tried as a divisor is the one used in the step just preceding.

To illustrate: Factor 1211. We try 2, 3, and 5 7)1211
by rules for divisibility; 7 is a factor, giving the 173
quotient 173. Since 1211 is a multiple of 173, no number
can be a factor of 173 which is not a factor of 1211. Hence
the first prime to try with 173 is 7 — the previous divisor.
By trial we find that 7, 11, and 13 will not work, and that
17 yields a quotient less than itself. Therefore 173 is
prime and the factors of 1211 are 7 and 173. In factoring
30330, 2 is used, 2 is tried again, 3 is used twice, 2 30330
3 is tried again, 5 is used, 5 is tried again, and 3 15165
finally the primes from 7 to 19 are tried before 3 5055
it is certain that 337 is prime. The answer may 5 1685
be stated in either of two ways: 30330 337
 $= 2 \times 3 \times 3 \times 5 \times 337$, or the prime factors of 30330
are 2, 3, 3, 5, 337.

The work may often be shortened by dividing by 4, by 9, or by 25, instead of using 2, 3, or 5 twice in succession. In the last example we might have seen that 9 is an exact divisor of 15165.

Common multiples by inspection. — As we have seen (p. 137), a multiple of a number is an exact container of it and a common multiple of two or more numbers is an exact container of all these numbers. The principal use of the process of finding common multiples is in connection with the addition and subtraction of fractions having unlike denominators, in which case it is necessary to

reduce them to fractions having the same denominator; thus, in adding $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, we might change them to 12ths, or to 24ths, or to 36ths. It is obvious that the work will be simplified by choosing the *least* common denominator. Since in most practical examples in adding and subtracting fractions we are dealing with small denominators, a systematic procedure for finding the least common multiple of the denominators by inspection is helpful.

The multiples of 4, of 6, and of 9 appear below so arranged as to show their positions in the natural number series.

4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
	6	12		18	24	30		36	42	48		54	60	
		9		18		27		36		45		54		

Observe that 12, 24, 36, are *common* multiples of the first two numbers and that 12 is their *least common multiple*. We see that the L.C.M. of 4 and 6 may be found by taking successive multiples of 4 until a multiple of 6 is found, or by taking successive multiples of 6 until a multiple of 4 is found. This may be done without writing the numbers, by merely counting by one number until a multiple of the other number is found; thus, 4, 8, 12. As 12 contains 6, 12 is the result desired. We also see that all the common multiples of 4 and 6 are multiples of 12; therefore the L.C.M. of 4, 6, and 9 must be the first multiple of 12 that contains 9. Taking successive multiples of 12 (12, 24, 36) until a multiple of 9 is found, we have 36, which is the L.C.M. of the three numbers. Hence the rule:

To find by inspection the L.C.M. of two numbers,

take successive multiples of one of the numbers until a multiple of the other is found. To find the L.C.M. of three or more numbers, find the L.C.M. of two of the numbers, find the L.C.M. of this result and a third number, etc. Thus, to find the L.C.M. of 12, 15, and 18, the *pupil's working explanation* would be: 15, 30, 45, 60; 60, 120, 180.

Lowest common multiple by factoring. — Occasionally the numbers are too large to handle by inspection. Find the L.C.M. of 210, 308, 462.

(1) $210 = 2 \times 3 \times 5 \times 7$	105	35	210
(2) $308 = 2 \qquad \qquad \times 7 \times 2 \times 11$	154	22	<u>22</u>
(3) $462 = 2 \times 3 \qquad \times 7 \qquad \times 11$	231	77	<u>42</u>
(4) L.C.M. = $2 \times 3 \times 5 \times 7 \times 2 \times 11 = 4620$			42 <u>4620</u>

Explanation :

1. Factor the first number as shown in (1). By rule, 2 is a factor, yielding the quotient 105 (quotients not easily carried in mind are written at the right). 3 is a factor of 105, giving the quotient 35. 35 gives the factors 5 and 7.

2. Factor the second number, trying in succession each factor of the first number before using others, and writing common factors of the two numbers in columns. 2 is a factor giving quotient 154; 3 and 5 are not factors; 7 is a factor giving quotient 22; 22 gives the factors 2 and 11 which are placed in new columns.

3. Factor the third number, first trying in succession the factors of the preceding numbers. We get the factors 2, 3, 7, and 11, which we place in their respective columns.

4. By definition, the L.C.M. must be a number which will contain all the given numbers, hence we take a factor from each column, whether complete or not, and find the product of these factors. The easiest way to get this product is to multiply the first number (210) by those factors of the L.C.M. not present in the first number, namely, 2 and 11.

Proof of the validity of the process. — The answer obtained is the L.C.M. required (1) if it contains each of the given numbers, and (2) if it would fail to contain every number were any of the factors omitted.

Since every factor of 210 is contained, 210 is contained; likewise for 308 and 462. Hence the answer obtained satisfies condition (1).

The omission of 11 from the answer would cause it not to contain 308, because 11 is a factor of 308. The omission of either 2 from the answer would cause it not to contain 308. The omission of the 7 from the answer would cause it not to contain any of the numbers. Were 5 omitted the answer could not contain 210; and if 3 were omitted the answer could not contain either 210 or 462. Hence the answer obtained satisfies condition (2).

Therefore the answer obtained is the L.C.M. required, and the method is a valid one for all examples of the sort.

Another method. — Most of the older textbooks give the following method for finding the L.C.M.

2	210	308	462
3	105	154	231
7	35	154	77
11	5	22	11
	5	2	1

This method of procedure has two serious disadvantages :
(1) Whenever an L.C.M. must be found in connection with the work with fractions it is also necessary to find how many times each of the given denominators is contained in the least common denominator, and this can be done much more easily by the first method, as will later appear.
(2) A process which, like this, will be very infrequently used will be better remembered, or more readily recalled, if its nature is understood ; and it is clear that in this respect the first method is decidedly superior. The authors do not favor the use of the second method.

To find the quotient when the L.C.M. is divided by any one of the given numbers, take the product of those factors of the L.C.M. which are not present in the given number ; thus, $4620 \div 308 = 3 \times 5 = 15$. These factors are very readily seen in the first method, but are discovered only with great difficulty in the second.

Highest common factor, or greatest common divisor. — Skill in seeing common factors is of frequent use in arithmetic, but there is no practical use for the process of finding the highest common factor. If the first method in the preceding section is taught, it takes only a few minutes to show the meaning of *highest common factor* and the method of finding it — take a factor from each column that is complete and find the product of these factors.

Cancellation. — There are a great many practical problems which involve no operations other than multiplication and division. The computations required in such problems may frequently be shortened by *expressing*, rather than performing, the operations and by *cancelling*

common factors from dividend and divisor. The principle involved is: *Dividing both dividend and divisor by the same number does not change the value of the quotient.* Success in this method depends upon the ability to discern the quotient relationship involved, and skill in seeing the common factors.

To illustrate: A writer has prepared 275 pages of typed copy for a book. The copy runs 28 lines of 12 words each per page. The book pages will run 33 lines of 10 words. How many pages will the book contain?

The number of pages in the book will be the quotient when the actual number of words is divided by the number of words on one page. The

2	5
6	55

$\times 275$, and the number of words

$$\frac{12 \times 28 \times 275}{10 \times 33} = 280$$

on each page is 10×33 . The

5	11
---	----

expression shows the first product

divided by the second; using the rules for divisibility, the common factors are readily seen and the answer obtained without recourse to written computation.

Cancellation is frequently useful in the solution of problems involving practical measurements, illustrations of which appear in the list of exercises.

HISTORICAL NOTES

Theory of numbers. — The early Greeks gave much attention to the study of the theory of numbers, and discovered many interesting properties. Some of these had no particular value, others led to further investigations which resulted in valuable discoveries. As explained on page 141, Eratosthenes investigated prime numbers

and invented the sieve method for determining whether a number was a prime.

Pythagoras, who was primarily a philosopher, was interested in the theory of numbers. He discussed polygonal numbers, ratio and proportion, factors of numbers, and numbers in series including arithmetical and geometrical progressions. He classified numbers with reference to the size of the number as compared with the sum of its integral divisors. The number was called *excessive*, *perfect*, or *defective* according as it was greater than, equal to, or less than, the sum of all its factors.¹ Thus, $28 = 1 + 2 + 4 + 7 + 14$. 28 is therefore a perfect number. Euclid showed that any number was a perfect number if of the form $2^{n-1} \times (2^n - 1)$ where $2^n - 1$ is a prime. Euclid also developed the theory of prime numbers and discussed the property of odd and even numbers and the arithmetical and geometrical progressions.

Amicable numbers, figurate numbers, polygonal numbers and pyramidal numbers were also defined and investigated by writers interested in this line of thought.²

Euclidean method of finding a highest common factor. — In the older arithmetics and algebras, methods for finding H.C.F. were carefully explained. One of the most interesting methods for finding H.C.F. is given by Euclid and is known as the Euclidean method. It is based upon the following principle: The greatest common divisor of two numbers is the greatest common divisor of the smaller

¹ BALL, W. W. R. — *A Short History of Mathematics*, p. 30.

² BROOKS, EDWARD. — *Philosophy of Arithmetic*, p. 385; Normal Publishing Company, Lancaster, Pa. (Out of print.)

For a recent investigation of amicable numbers see MASON, THOMAS E. — *American Mathematical Monthly*, XXVIII, No. V.

and of the remainder found by dividing the greater by the smaller.

The application of the principle to an example is an interesting procedure. Let us find the H.C.F. of 319 and 493 by this method :

$$\begin{array}{r}
 319)493(1 \\
 \underline{319} \\
 174)319(1 \\
 \underline{174} \\
 145)174(1 \\
 \underline{145} \\
 29)145(5 \\
 \underline{145} \\
 0
 \end{array}$$

The H.C.F. of 319 and 493 is the H.C.F. of 319 and 174; the H.C.F. of 319 and 174 is the H.C.F. of 174 and 145, which in turn is the H.C.F. of 145 and 29. This is 29. Therefore 29 is the H.C.F. of 319 and 493.

The subject of finding H.C.F. is practically obsolete, as no real need exists for its use.

EXERCISES

1. Illustrate *factor, multiple, common multiple, least common multiple, common divisor, greatest common divisor, prime number, composite number*.

2. Name the common multiples of 6 and 9, their lowest common multiple; the common multiples of 4, 6, and 9, their lowest common multiple.

3. Mention all the exact divisors of 42; all possible pairs of factors; the prime factors.

4. What are the common factors and the highest common factors of (a) 18 and 24, (b) 16, 24, and 40?

5. By the *sieve* method find the list of prime numbers to 100.

6. List the composites to 100 and factor each mentally into all possible pairs of factors.

7. Using the rules for divisibility by 2, 4, 8, 5, 25, 125, 3, 9, find by inspection which of these numbers are exact divisors of 612, 2328, 13695, 33150, 3756.

8. Determine whether 961 is prime and explain your procedure. How does this procedure relate to the sieve method?

9. Resolve the following numbers into their prime factors, and explain the process. If any number is prime, state the largest divisor tried. 961, 241, 301, 126, 156, 171, 243, 273, 1441, 751, 2613, 323, 1774, 3774.

10. Give the working explanation for finding by inspection the L.C.M. of :

- | | | |
|---------------|---------------|---------------------------|
| (a) 4, 6, 8 | (f) 6, 12, 9 | (k) 15, 25, 20 |
| (b) 4, 6, 10 | (g) 6, 9, 18 | (l) 4, 6, 8, 9, 12, 15 |
| (c) 6, 8, 9 | (h) 20, 25, 8 | (m) 3, 8, 12, 9, 6, 4, 18 |
| (d) 8, 12, 15 | (i) 7, 14, 21 | (n) 24, 9, 8, 36, 18, 12 |
| (e) 8, 12, 18 | (j) 8, 16, 24 | (o) 21, 28, 12, 9 |

11. Find (a) by the factoring method the L.C.M. of the following sets of numbers, and (b) the quotients of the L.C.M. divided by each of the given numbers taken separately.

- | | |
|--------------------|-------------------|
| (a) 660, 1170, 858 | (c) 330, 140, 231 |
| (b) 126, 238, 1365 | (d) 57, 95, 133 |

12. To multiply $12 \times 3 \times 6$ by 3, how many of the factors must be multiplied? How many must be divided to divide $12 \times 3 \times 6$ by 3?

13. State under what conditions a number is divisible by 15, by 20, by 36.

14. Explain the relative merits of the two written methods of finding the L.C.M. of numbers.

15. Is cancellation a principle or a process? Explain.

16. Explain the mathematical connection between the *sieve* of Eratosthenes and the rule for factoring a number.

17. In what respect is the written process of finding L.C.M. like the method of inspection? HINT. Use both methods on an easy example.

18. Use cancellation in solving the following problems :

(a) There are 2240 lb. in a long ton. How many short tons in 8 long tons?

(b) How many long tons in 8 short tons?

(c) A man gets a salary of \$332 per month. If he works 308 days in a year, what does he earn by a day of labor?

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(d) I have floor space 8' by 6' for a coal bin. How deep must I make my bin to hold 9 tons of coal, allowing 38 cu. ft. for a ton?

(e) A contractor finds that it requires the labor of 12 men working 8 hours a day for 25 days to do half of a ditching contract. How many men will he require to finish the job in 18 days if they work 2 hours a day overtime?

19. State the sum by factors: 5×8 and 7×8 ; $6 \times 3 \times 4$ and $6 \times 3 \times 5$; $8 \times 5 \times 6$ and $6 \times 5 \times 9$.

20. Find the results by cancellation and state the rules for divisibility used:

$$(a) \frac{33 \times 45 \times 95 \times 43}{129 \times 39 \times 19 \times 15} \qquad (b) \frac{237 \times 847 \times 125}{231 \times 75 \times 79}$$

$$(c) \frac{18 \times 56 \times 91 \times 47 \times 38 \times 81 \times 132}{44 \times 57 \times 241 \times 13 \times 75}$$

21. Discover deductively a rule for exact divisibility by 11. **HINT.** $3957 = 7 + 50 + 900 + 3000$.

1 = No. 11's + 1	7 = No. 11's + 7
10 = No. 11's - 1	50 = No. 11's - 5
100 = No. 11's + 1	900 = No. 11's + 9
1000 = No. 11's - 1	3000 = No. 11's - 3
	3957 = No. 11's + (7 + 9) - (5 + 3)

22. Find two numbers whose H.C.F. is 12, and whose L.C.M. is 924.

23. Reduce to lowest terms using the rule: Divide both terms by successive common factors until no one of 2, 3, 5, 7, or 11 is a common factor. Then find the prime factors of *one* of the terms and test the other by these primes.

$\frac{1326}{1938}$	$\frac{471}{628}$	$\frac{735}{1715}$	$\frac{3158}{3498}$	$\frac{1794}{1976}$
---------------------	-------------------	--------------------	---------------------	---------------------

24. By experiment show that probably every prime number is one more or one less than a multiple of 6.

25. By deduction, prove the statement in Exercise 24 as absolute. **HINT.** $6n, 6n + 1, 6n + 2, 6n + 3, 6n + 4, 6n + 5$. Every integer is found in the series above by the substitution of 0, 1, 2, 3 . . . for n .

$6n$ is a composite number because, . . . $6n - 1$ may be a prime because, . . . $6n - 2$ is a composite because, . . . etc.

26. Divide 47956 by 7777, obtaining exact quotient in simplest fractional form.

27. Establish deductively the rule for exact divisibility of a number by 4. HINT. Take 14788 and separate it into $147 \times 100 + 88$. Why is the first part of the number exactly divisible by 4? Why is the sum of the two parts a multiple of 4?

28. Establish deductively the rule for exact divisibility of a number by 8. HINT. Take a number in which the number denoted by the three right-hand digits is a multiple of 8, as 43584, and separate it into two parts, thus: $43 \times 1000 + 584$.

METHODS OF TEACHING

Practical values. — School arithmetics of a generation ago usually devoted a chapter to *factoring*, *lowest common multiple*, and *highest common factor*. In many modern textbooks no such chapter is to be found. Those parts of that chapter which children need in their work with fractions are taught in connection with fractions. The cancellation method in the solution of problems (p. 147) is well worth teaching in the higher grades, but it is not indispensable. Our one problem in the paragraphs which follow is to suggest teaching procedures for the development of those skills which make for success in the work with fractions.

Factoring. — *Pairs of factors.* — Skill in factoring may be secured by drill exercises in declaring all possible pairs of factors for composite numbers up to 144. This work should be oral, or mental. "Flash cards" bearing the composite numbers are a good device; as 30 is shown the children respond, 2×15 , 3×10 , 5×6 ; as 60 is shown the children say 2×30 , 3×20 , 4×15 , 5×12 , 6×10 .

Cancellation. — It is important that children should study the subject of cancellation in problem situations which permit of the use of the process; otherwise whatever mechanical skill may be developed is almost sure to go unused. However, it may happen that the children, after solving several problems, discover that lack of skill in seeing factors is interfering with their success. Then is the time when abstract drill exercises can be worked with profit and with enthusiasm; but let the work *begin and end with practical problems.*

Rules for divisibility. — To teach any of the rules inductively the following steps may be taken :

1. *State the aim of the lesson.* Let us see if we can find a quick way of telling whether a number is divisible by 3.

2. *Obtain some numbers that are divisible by the number in question.* Each of you take a number of two or more figures and multiply it by 3. You know that the results are divisible by 3.

3. *Search for some common characteristic of these results.* Add the digits of your numbers which are divisible by 3 and see if there is anything in which they agree. Have children report and place the numbers in one column and the sums of their digits in another column on the board. Study the second column.

4. *Draw tentative conclusion.* A number that is divisible by 3 has the sum of its digits divisible by 3.

5. *Study the original numbers.* Take your original numbers and find the sums of their digits. Which sums are divisible by 3? See if the numbers are divisible by 3.

6. *Make the final conclusion.* What is true of numbers that are divisible by 3? Of those that are not divisible by 3? Tell in your own words how to tell, without dividing, whether a number is divisible by 3. Here is a short statement of the rule; learn it: A number is divisible by 3 if the sum of its digits is divisible by 3.

The technique of teaching certain other topics in this chapter is discussed in the chapter on fractions.

EXERCISES

1. (a) List the topics treated in this chapter which might properly be taught for their utilitarian value, and state what value each topic possesses.

(b) List any additional topics which you would desire to teach for their mathematical interest to children. For what grade would this work be suited?

2. State the steps you would take in developing inductively the rule for divisibility by 4, by 9, by 5.

3. Explain how you would use the array of numbers on page 144 in teaching the meaning of *multiple*, *common multiple*, and *lowest common multiple*.

4. Prepare a list of numbers to be used for practice on the rules for divisibility.

5. Prepare a list of fractions to be reduced to lowest terms (a) in which common factors of the terms are evident; (b) in which common factors are not evident.

6. Explain how you would teach (b) above.

7. Show how you would discover inductively the principle involved in cancellation.

CHAPTER VII

FRACTIONS

TEACHER'S KNOWLEDGE

HISTORICAL NOTES

Fractions among the ancients. — The need of expressing a part of a whole gave rise to a new problem for the world to wrestle with, and that it proved to be a knotty one is evidenced by the fact that many different ways of meeting this need were found by different peoples. The earliest answers agreed in this particular — the whole was thought of as *broken up* into a number of equal parts ; the resulting expression came therefore to be called a fraction (*fractus*, broken).

The Egyptians used unit fractions. In one of the oldest mathematical manuscripts, written by Ahmes¹ about 1550 B.C. — now in the British Museum and called The Rhind Papyrus, for the man who purchased it from Egypt — much space is devoted to unit fractions. Thus $\frac{5}{12}$ is thought of as $\frac{1}{3} + \frac{1}{12}$. The Babylonians, on the other hand, varied the numerator but used a constant denominator. They considered the whole as broken up into 60 equal parts and each of these in turn as subdivided into 60 equal parts. These are called *sexagesimal* fractions. $\frac{5}{12}$ was thought of as $\frac{25}{60}$. Since the

¹ SMITH, D. E. — *History of Mathematics*, p. 47.

denominator was constant, they expressed it by accent marks — a single accent (') for the first power of 60; a double accent (") for 60^2 or 3600, etc. Thus $\frac{25}{60}$ might be written as $25'$; $\frac{1}{8}$, which equals $\frac{7}{60} + \frac{30}{3600}$, as $7' 30''$. The Romans divided the whole into 12 equal parts; in their abacal reckoning, buttons were used to represent $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{72}$, respectively.¹

DEVELOPMENT OF SUBJECT MATTER

Various concepts of fractions. — The most vivid concept of a fraction arises from the concrete picture of a whole broken up into a number of equal parts.

Thus: AC is 3 of the 5 equal parts of AB , or three-fifths of AB .

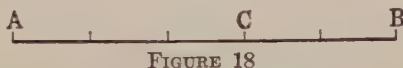


FIGURE 18

A fraction may also express a quotient or ratio. Let us study some concrete illustrations of these important concepts of a fraction.

Three-fourths of a dollar can be obtained by changing the dollar into quarters and taking three of them, or by changing three dollars into quarters and taking one-fourth of these. We may illustrate the same thought by means of the line diagram :

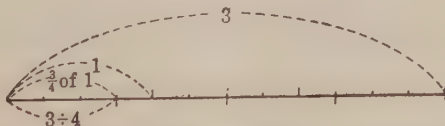


FIGURE 19

¹ CAJORI, F. — *History of Elementary Mathematics*, p. 38.

In both illustrations we see that three-fourths means three of the four equal parts of one, or one of the four equal parts of three. That is, $\frac{3}{4} = 3 \div 4$.

If John is 9 years old and Henry is 12 years old, we may say that John is three-fourths as old as Henry. This situation illustrates the ratio concept. The two quantities, 9 years and 12 years, are separated into equal parts in such a way that this equal part, 3 years, is exactly contained in each of the quantities. It is contained in 9 years 3 times and in 12 years 4 times; hence 9 is 3 of the 4 equal parts of 12, or $\frac{3}{4}$ of 12.

Terms of a fraction. — To express a fraction, two numbers are necessary, one to tell into how many equal parts the whole has been divided, and the other to state how many of these parts are considered. The *denominator* shows into how many equal parts the whole has been divided. The *numerator* shows how many of the equal parts are chosen. These are called the *terms* of a fraction. To write a fraction, place the numerator above the denominator with a short horizontal line between, thus, $\frac{3}{5}$. Sometimes an oblique instead of a horizontal line is used; thus, $3/5$. To read a fraction read the number of equal parts considered, *i.e.*, the numerator (*numerare*, to number), and then call the name of each equal part, which is revealed by the denominator (*de-nominare*, to designate by name). Thus, $\frac{5}{11}$ is read *five-elevenths*. We may also read a fraction as an expression of division. $\frac{3}{4}$ may be read, *3 divided by 4*; $\frac{a}{b}$ is read *a divided by b*.

Kinds of fractions. — Since fractions may be classified in various ways, we may consider (1) the possibility of

expressing an integer with a fraction. An integer plus a fraction may be written by omitting the plus sign. Thus $4 + \frac{2}{3}$ may be written $4\frac{2}{3}$. This is called a *mixed number*.

(2) We may consider the relative size of the numerator and denominator. The numerator of a fraction may be equal to, greater than, or less than the denominator. A *proper fraction* is a fraction in which the numerator is less than the denominator. An *improper fraction* is a fraction in which the numerator is equal to or greater than the denominator. (3) We may consider the possibility of having a fraction in the *terms*. There may be a fraction in one term only, in both terms, or in neither term, as $\frac{\frac{4}{5}}{5}$, $\frac{4}{\frac{5}{6}}$, $\frac{\frac{4}{5}}{\frac{5}{6}}$, $\frac{4}{5}$; the first three are called *complex fractions*; the last is called a *simple fraction*. This is really an expres-

sion of division. Hence $\frac{\frac{4}{5}}{\frac{5}{6}}$ is read $\frac{4}{5}$ divided by $\frac{5}{6}$.

The multiplication of two or more fractions may be indicated by writing *of* instead of ($\times$). Thus $\frac{3}{4} \times \frac{5}{8}$ may be written $\frac{3}{4}$ of $\frac{5}{8}$. Such an expression is sometimes called a *compound fraction*.

All the operations with or upon fractions may be explained in the light of a few general principles, together with the definitions previously given. Six of these principles have to do with the effects upon the value of a fraction when one or both terms have been multiplied or divided by an integer. These are known as the six cardinal or fundamental principles, and are developed in the following paragraphs. They form the logical basis for reduction to higher and lower terms, reductions which are employed in addition and subtraction of fractions.

Other principles are developed in the discussions of multiplication and division of fractions.

Multiplying the numerator. — Let us discover the effect upon a fraction of multiplying its numerator.

Induction

Let us take the fraction $\frac{2}{9}$ and multiply the numerator by 2, obtaining $\frac{4}{9}$; also the fraction $\frac{3}{10}$ and multiply the numerator by 3, obtaining $\frac{9}{10}$.

Deduction

Multiplying the numerator multiplies the number of equal parts taken without changing the size of the parts, because the numerator indicates the number

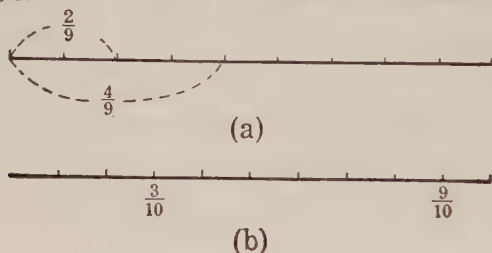


FIGURE 20

Examining the above diagrams we discover that $\frac{4}{9}$ is 2 times $\frac{2}{9}$, and that $\frac{9}{10}$ is 3 times $\frac{3}{10}$.

Therefore it is probably true that multiplying the numerator multiplies the fraction.

of equal parts considered.

This multiplies the whole fraction, because the greater the number of like parts in the group, the greater the group. Hence:

Principle I. — Multiplying the numerator multiplies the fraction.

Multiplying the denominator. — Consider the effect upon a fraction of multiplying the denominator.

Induction

Let us take the fraction $\frac{2}{9}$ and multiply the denominator by 3,

Deduction

Multiplying the denominator of a fraction multiplies the num-

obtaining $\frac{2}{9}$; also take $\frac{2}{3}$ and multiply the denominator by 2, obtaining $\frac{2}{6}$.
 ber of equal parts into which the whole has been divided without affecting the number of parts

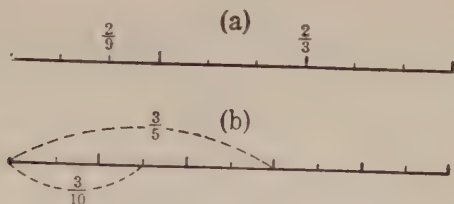


FIGURE 21

Examining the diagrams we discover that $\frac{2}{9}$ is $\frac{1}{3}$ of $\frac{2}{3}$; and that $\frac{3}{10}$ is $\frac{1}{2}$ of $\frac{3}{5}$.

Therefore it is probably true that multiplying the denominator divides the fraction.

taken, because the denominator shows into how many equal parts a whole has been divided. This divides the size of each part, because the greater the number of parts in the whole the smaller the size of each.

Dividing the size of each part divides the whole fraction, because the smaller the size of each part in a given number of parts the smaller the quantity. Hence:

Principle II.—Multiplying the denominator of a fraction divides the fraction.

Multiplying both terms. — Consider the effect of multiplying both terms of a fraction by the same number.

Induction

Let us take the fraction $\frac{2}{3}$ and multiply both terms by 3, obtaining $\frac{6}{9}$; also take $\frac{2}{3}$ and multiply both terms by 2, obtaining $\frac{4}{6}$.

Deduction

Multiplying the numerator and denominator of a fraction by the same number first multiplies and then divides the fraction by that

Examining the diagrams we discover that $\frac{2}{3} = \frac{4}{6}$, and $\frac{3}{5} = \frac{6}{10}$. number, because multiplying the numerator multiplies the fraction

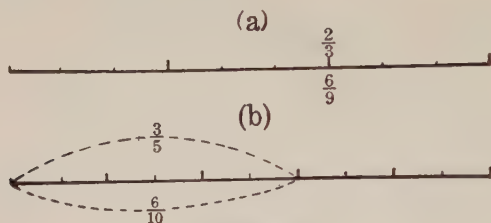


FIGURE 22

Therefore it is probably true that multiplying both terms of a fraction by the same number does not change the value of the fraction.

and multiplying the denominator divides the fraction.

Multiplying and dividing a fraction by the same number does not change the value of the fraction, because the operations of multiplication and division cancel. Hence:

Principle III. — Multiplying both terms of a fraction by the same number does not change the value of the fraction.

The six principles summarized. — The three remaining principles have to do with the effect upon the value of a fraction of *dividing* the numerator, *dividing* the denominator, and *dividing* both terms by the same number. Their inductive and deductive developments are similar to those given above. Thus, we have the six so-called *cardinal* or *fundamental* principles of fractions.

- I. Multiplying the numerator multiplies the fraction.*
- II. Multiplying the denominator divides the fraction.*
- III. Multiplying both terms by the same number does not change the value of the fraction.*
- IV. Dividing the numerator divides the fraction.*
- V. Dividing the denominator multiplies the fraction.*
- VI. Dividing both terms by the same number does not change the value of the fraction.*

Meaning of reduction. — In performing operations with fractions it is often necessary or convenient to change their form. The process of changing the form without altering the value is known as *reduction*. The useful forms of reduction of fractions will now be explained in the light of the foregoing definitions and principles.

Reducing mixed numbers to improper fractions. — Let us reduce $4\frac{2}{3}$ to fractional form. $4\frac{2}{3}$ means 4 plus $\frac{2}{3}$. Since in *one* there are 3 thirds, in *four* there are 4 times 3 thirds or 12 thirds; $\frac{12}{3} + \frac{2}{3} = \frac{14}{3}$; hence $4\frac{2}{3} = \frac{14}{3}$.

Reducing improper fractions to mixed numbers. — Let us reduce $\frac{16}{3}$ to a mixed number. $\frac{16}{3}$ means $16 \div 3$ (see page 158). Performing the division, we obtain $5\frac{1}{3}$. Or, we may reason thus: Since in 1 unit there are 3 thirds, there are as many units in 16 thirds as the times 3 is contained in 16, or $5\frac{1}{3}$.

Reducing fractions to higher terms. — Let us reduce $\frac{5}{9}$ to 36ths. To get the denominator 36 we shall have to multiply the given denominator by 4. Multiplying both terms by 4, we obtain $\frac{20}{36}$; therefore $\frac{5}{9} = \frac{20}{36}$, because multiplying both terms by the same number does not change the value of the fraction.

Reducing fractions to lowest terms. — Let us reduce $\frac{36}{54}$ to lowest terms. Since dividing both terms of a fraction by the same number does not change the value of the fraction, we divide by any common factor which is evident, treat the result in the same way, and so proceed until the terms are prime to each other.

E.g., $\frac{36}{54} = \frac{6}{9} = \frac{2}{3}$ (dividing both terms by 6 and then by 3).

NOTE. — The larger the number by which we divide, the fewer the divisions necessary. Contrast dividing at once by 18 with dividing by 2, by 3, and by 3.

$$\frac{36}{54} = \frac{2}{3} \qquad \frac{36}{54} = \frac{18}{27} = \frac{6}{9} = \frac{2}{3}$$

When no common factor is evident at a glance, it is convenient to factor one of the terms, to determine which of these prime factors are contained in the other term, and to divide both terms by such factors. Illustration: Suppose 204 pupils were present in a school of 323; it is possible that the fraction $\frac{204}{323}$ may be reduced to lowest terms. Let us try it and see. Since no common factor is readily seen, we factor 204. $204 = 2 \times 2 \times 3 \times 17$. Neither 2 nor 3 are factors of 323, but 17 is. Dividing both terms by 17, we find that $\frac{204}{323} = \frac{12}{19}$.

NOTE. — In such cases, the usual procedure would be to reduce the fraction to a decimal (see page 211).

Reducing complex fractions to simple.¹ — *First form.* — Simplify $\frac{\frac{2}{3}}{\frac{8}{9}}$. Since this means $\frac{2}{3} \div \frac{8}{9}$, we may apply the rule for dividing a fraction by a fraction (see page 172).

¹ So far as arithmetic in the grades is concerned the subject of complex fractions is practically obsolete.

$$\frac{2}{3} \div \frac{8}{9} = \frac{2}{3} \times \frac{\overset{3}{9}}{\underset{4}{8}} = \frac{3}{4}$$

Second form. — Since multiplying both terms of a fraction by the same number does not change the value of the fraction, we may multiply both terms by 9, the L.C.M. of the denominators :

$$\frac{\overset{2}{\cancel{2}} \times \overset{3}{\cancel{9}}}{\underset{8}{\cancel{8}} \times \underset{9}{\cancel{9}}} = \frac{6}{8} = \frac{3}{4}$$

Addition. — The process of addition depends upon the same general principles that govern the addition of integers. Let us add $1\frac{5}{9}$, $2\frac{3}{4}$, and $\frac{2}{3}$.

Full explanation. — Since only like numbers, or those which can be reduced to the same denomination, can be added, we must reduce these fractions to equivalent fractions of higher terms, having a common denominator. Since the denominators are small, the least common denominator should be found by inspection. See page 144. The L.C.M. of 9, 4, 3, is 36. Reducing each fraction to 36ths we obtain $\frac{20}{36}$, $\frac{27}{36}$, and $\frac{8}{36}$; the sum of the 36ths is $\frac{71}{36}$ or $1\frac{35}{36}$; we write the $\frac{35}{36}$ in the fractional column and carry the 1 to units' column. The sum of the units is 4.

NOTE. — It is better not to write each denominator in the right-hand column, because (1) it is understood, since the L.C.D. (lowest common denominator) is written below the line; (2) it saves time; and (3) it is easier to add the numerators when standing alone.

A more difficult type. — Add $5\frac{11}{18}$, $3\frac{17}{24}$, $2\frac{13}{36}$. Here the L.C.D. may be found by factoring more readily than by inspection (see page 145).

$5\frac{1}{8}$	220	$18 = 2 \times 3 \times 3$
$3\frac{1}{2}$	255	$24 = 2 \times 3 \times 2 \times 2$
$2\frac{3}{5}$	156	$30 = 2 \times 3 \times 5$
$11\frac{2}{3}$	$\frac{2}{3}$	$\times 5$
$11\frac{2}{3}$	$\frac{2}{3}$	L.C.D. = $2 \times 3 \times 3 \times 2 \times 2 \times 5 = 360$
	17	13
	15	12
	85	$360 \overline{)631}$
	17	360
	255	271

The explanation is the same as for the preceding example. The quotient of the L.C.D. by each denominator may be found by use of factors without performing the actual division (see page 147). There is little, if any, practical need for adding fractions with large denominators; in such cases it is usually much more convenient to reduce the fractions to decimals and then add.

Subtraction. — The same general principles apply as in subtraction of integers and any one of the three methods — the first Italian, second Italian, or Austrian — may be employed. Let us subtract $3\frac{3}{4}$ from $6\frac{2}{3}$, using the Austrian method. A convenient form of arranging appears below.

Full explanation. — Since only numbers of like denominations can be subtracted, one must reduce the fractions to equivalent fractions having a common denominator. The L.C.D. is 12: $\frac{2}{3} = \frac{8}{12}$. $\frac{3}{4} = \frac{9}{12}$. To $3\frac{3}{4}$ enough must be added to give $6\frac{2}{3}$ or to $\frac{9}{12}$ enough must be added to give a number having $\frac{8}{12}$ in its fractional column, *i.e.*, enough must be added to $\frac{9}{12}$ to give $1\frac{8}{12}$ or $\frac{20}{12}$. $\frac{9}{12}$ and $\frac{11}{12}$ are $\frac{20}{12}$. Write $\frac{11}{12}$ in fractional column. When $\frac{11}{12}$ is added to $3\frac{3}{4}$, we have 4 and $\frac{2}{3}$; we must now find what must be added to 4 to give 6; 2 must be added; we write 2 in units' column.

Working explanation. — $\frac{2}{3}$ and $\frac{11}{12}$ are $1\frac{8}{12}$ or $\frac{20}{12}$; 4 and 2 are 6.

Multiplication. — Let us consider the meaning of multiplying by a fraction. The multiplier, when an integer, shows how many times the multiplicand is to be taken as an addend. Since it is impossible to use a number “ $\frac{2}{3}$ of a time” or “ $\frac{3}{4}$ of a time,” etc., we extend our notion of multiplication and consider taking the indicated fractional *part* of the multiplicand (see page 77). Thus, we may take $\frac{2}{3}$ of any number; that is, we may find $\frac{1}{3}$ of it by dividing it into 3 equal parts, and take this result twice.

Various types of examples arise in multiplication of fractions, depending upon the term in which the fraction occurs. We may have a fraction in the multiplier, a fraction in the multiplicand, or a fraction in both terms. Let us consider each type separately.

(a) *To multiply an integer by a fraction.* — To find $\frac{3}{4}$ of 12, we divide 12 into 4 equal parts ($12 \div 4$), thus finding one fourth of 12, and multiply this result by 3, $(12 \div 4) \times 3$. Since the result will be the same whether the division or the multiplication is performed first, we may proceed according to convenience. In finding $\frac{3}{4}$ of 12, it is more convenient to find $\frac{1}{4}$ of 12 first and then multiply this result by 3. In finding $\frac{3}{4}$ of 15, it is more convenient to multiply first by 3 and then divide the result by 4. $3 \times 15 = 45$; $45 \div 4 = 11\frac{1}{4}$.

For larger numbers where written computation is needed the work may be arranged as follows:

$$\frac{3}{4} \times \overset{69}{276} = 207 \qquad \frac{3}{4} \times 275 = \frac{825}{4} = 206\frac{1}{4}$$

(b) *To multiply a fraction by an integer.* — Let us consider $\frac{3}{8}$ multiplied by 2, and $\frac{2}{9}$ multiplied by 3. $\frac{3}{8}$ multi-

plied by 2 means $\frac{3}{8} + \frac{3}{8}$, which equals $\frac{6}{8}$ or $\frac{3}{4}$. $\frac{2}{9}$ multiplied by 3 equals $\frac{2}{9} + \frac{2}{9} + \frac{2}{9}$, which equals $\frac{6}{9}$ or $\frac{2}{3}$. In each case the first form of the result might have been obtained by multiplying the numerator and the second form, by dividing the denominator. Hence the probable rule is: To multiply a fraction by an integer, either multiply the numerator or divide the denominator. This rule is convenient for mental multiplication. It is seen to be really a combination of two of the six fundamental principles united to form a single rule: *Dividing the denominator multiplies a fraction; and multiplying the numerator multiplies a fraction.*

Multiply $\frac{1}{4}$ by 4. Dividing the denominator, we obtain $\frac{1}{1}$ or 1. Multiply $\frac{1}{4}$ by 5. Since the denominator is not exactly divisible by 5, we multiply the numerator and obtain $\frac{5}{4}$ or $1\frac{1}{4}$. This rule, as stated above, is particularly helpful in *mental* work. The written work may take the form of written work in (a) or (c) or (d).

(c) *To multiply a fraction by a fraction. —*

Induction

Let us take $\frac{2}{3}$ of $\frac{3}{4}$ and $\frac{2}{3}$ of $\frac{3}{8}$.

Deduction

Let us take $\frac{2}{3}$ of $\frac{3}{4}$. $\frac{2}{3}$ of $\frac{3}{4}$ is 2

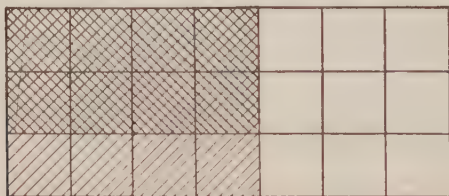


FIGURE 23

The double shaded portion is $\frac{2}{3}$ of $\frac{3}{4}$ of the whole rectangle, or $\frac{6}{12}$ of it.

of the 3 equal parts into which $\frac{3}{4}$ has been divided, since the denominator shows the number of

Also, 2 of the 3 equal parts of $\frac{2}{3}$ of the line is 4 of the 15 equal parts of the whole line, or $\frac{4}{15}$ of it.

parts into which a whole has been divided, and the numerator shows the number of such parts that are chosen. This would be expressed



FIGURE 24

Examining the above results, we discover that in each case the numerator of the answer is the product of the two numerators and the denominator of the answer is the product of the two denominators.

Therefore, it is probably true that to multiply a fraction by a fraction, multiply the numerators for the new numerator and the denominators for the new denominator.

Multiply $\frac{5}{6}$ by $\frac{12}{25}$. Applying the rule, we should obtain $\frac{5 \times 12}{6 \times 25}$ or $\frac{60}{150} = \frac{2}{5}$. Instead of performing the multiplication and then dividing both terms by the same number it is customary to divide both terms by the same number before performing the multiplication; we may do this

as $(\frac{5}{6} \div 3) \times 2$. Taking 1 of the 3 equal parts of $\frac{5}{6}$, *i.e.*, finding the value of $\frac{5}{6} \div 3$, we have $\frac{5}{18}$, because to divide a fraction by an integer we multiply the denominator.

Since one of the 3 equal parts is $\frac{5}{18}$, 2 of the 3 equal parts is 2 times $\frac{5}{18}$, or $\frac{10}{18}$, because to multiply a fraction by an integer we multiply the numerator.

$$\therefore \frac{5}{6} \times \frac{12}{25} = \frac{10}{25}.$$

Therefore, to multiply a fraction by a fraction, multiply the numerators for the new numerator and the denominators for the new denominator.

because the product is not affected by the order of the factors. The work appears as at the left.

Hence, for practical purposes, to multiply a fraction by a fraction, multiply the numerators for a new numerator and the denominators for a new denominator, *cancelling when possible*.

(d) *Multiplication involving mixed numbers*. — Such examples may be performed by reducing the mixed numbers to improper fractions and proceeding as in the general cases (a), (b), or (c), or the multiplication may be performed without reducing the mixed numbers. Illustrations follow.

Multiply $7\frac{5}{8}$ by 6.

$\frac{7}{6} \times \frac{12}{25} = \frac{2}{5}$ We first multiply the fraction $\frac{5}{8}$ by 6 by multiplying the numerator, then we multiply the integer, adding the first result.

Brief working explanation: $\frac{30}{8}$, or $3\frac{3}{4}$; 42, 45.

$$\frac{3}{4} \times \frac{9}{5} = 27$$

Multiply $3\frac{3}{4}$ by $7\frac{1}{2}$. Reducing both mixed numbers to improper fractions, we obtain $\frac{15}{4}$ and $\frac{15}{2}$; we then apply the general rule for multiplying a fraction by a fraction.

Multiply $52\frac{5}{8}$ by $38\frac{3}{8}$. Here it is convenient to multiply the mixed numbers without reducing to improper fractions, because the denominators are small and the integers are numbers of two or more figures.

Explanation: $52\frac{5}{8} = 52 + \frac{5}{8}$; $38\frac{3}{8} = 38 + \frac{3}{8}$. The multiplication of two mixed numbers may be regarded as the multiplication of two binomials, $(a + b)(c + d)$, or the order of procedure may be related to an example in the multiplication of integers. $\frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$; $\frac{3}{8} \times 52 = \frac{156}{8}$; we write 156 to the right and divide it by 8, setting the result as the second partial product; $38 \times \frac{5}{8} = \frac{190}{8}$; we write 190 at the right and divide it by 8, writing the result as the third partial product. We then multiply 52 by 38 and add all the partial products. The work should appear as at the left.

$$\begin{array}{r} 52\frac{5}{8} \quad 156 \\ 38\frac{3}{8} \quad 190 \\ \hline \frac{1}{8} \quad 15 \\ 31\frac{1}{8} \quad 6 \\ 31\frac{3}{8} \quad 20 \\ 416 \\ 156 \\ \hline \text{Ans. } 2039\frac{15}{64} \end{array}$$

Division. — The same types of examples arise as in multiplication, but for practical purposes the division of an integer by a fraction may be treated as a special case of dividing a fraction by a fraction.

(a) *Division of a fraction by an integer.* — Principles II and IV (p. 163) may be used. Where possible, as in $\frac{4}{5} \div 2$, we divide the numerator (Principle IV), obtaining $\frac{2}{5}$; when the numerator is not exactly divisible by the integer, we multiply the denominator, for example: $\frac{4}{5} \div 3 = \frac{4}{15}$. The diagrams show the two methods objectively.

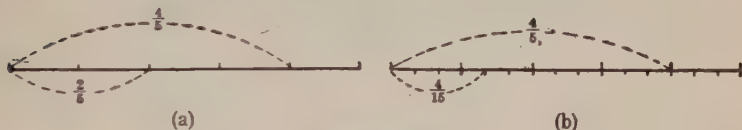


FIGURE 25

(b) *Division of a fraction by a fraction.*

Induction

Let us take $\frac{2}{3} \div \frac{5}{7}$, and $\frac{3}{4} \div \frac{4}{5}$. We must find how many times $\frac{5}{7}$ of a thing contains $\frac{2}{3}$ of the same thing.

Deduction

Let us take $\frac{2}{3} \div \frac{5}{7}$. We may multiply the divisor by something that will make it an integer, which will give us the



FIGURE 26

$\frac{2}{3} = 14$ small parts.

$\frac{5}{7} = 15$ small parts (same size).

That is, $\frac{2}{3} \div \frac{5}{7} = 14 \div 15 = \frac{14}{15}$.

$\frac{3}{4} = 15$ small parts (Fig. 27).

case of a fraction divided by an integer. We must then multiply the dividend by the same number.

$$\frac{2}{3} \div \frac{5}{7} = (\frac{2}{3} \times \frac{7}{5}) \div (\frac{5}{7} \times \frac{7}{5})$$

$\frac{4}{8} = 16$ small parts (same size).

$$\frac{3}{4} \div \frac{5}{8} = 15 \div 16 = \frac{15}{16}.$$

In each case the same answer might have been obtained by inverting the divisor and multiplying; $\frac{3}{4} \times \frac{7}{8} = \frac{14}{16}$ and $\frac{3}{4} \times \frac{5}{4} = \frac{15}{16}$.

because multiplying both dividend and divisor by the same number does not change the value of the quotient.

$$(\frac{3}{4} \times \frac{7}{8}) \div (\frac{5}{7} \times \frac{7}{8}) = (\frac{3}{4} \times \frac{7}{8}) \div 1$$

since $\frac{5}{7} \times \frac{7}{8} = 1$, by rule for mul-

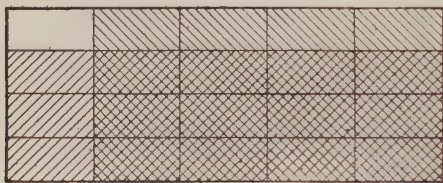


FIGURE 27

Therefore, it is probably true that to divide a fraction by a fraction, invert the divisor and multiply.

tiplying a fraction by a fraction.

$$(\frac{3}{4} \times \frac{7}{8}) \div 1 = \frac{3}{4} \times \frac{7}{8}$$

because any quantity divided by 1 equals itself.

$$\text{Therefore } \frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{8}.$$

Hence the rule: To divide a fraction by a fraction, invert the terms of the divisor and multiply.

A second deductive method is to reduce both fractions to equivalent fractions having their L.C.D., and divide the numerator of the dividend by the numerator of the divisor, because this is a case of quotition or measurement division, that is, finding how many times one quantity contains another of the same kind.

$$\text{Thus, } \frac{3}{4} \div \frac{5}{7} = \frac{14}{16} \div \frac{15}{16} = \frac{14}{16}.$$

Hence the rule.

(c) *Division involving mixed numbers.* — The mixed numbers may be reduced to improper fractions and the general rules above may then be applied, or they may be retained

as mixed numbers and the division performed as in the illustrations below.

(1) Divide $114\frac{5}{8}$ by 8.

8) $\overline{114\frac{5}{8}}$ *Full or Teacher's explanation:* 11 tens $\div 8 = 1$ ten with 3 tens remaining; write the 1 in tens' place; 3 tens or 30 units and 4 units are 34 units; $34 \text{ units} \div 8 = 4$ units with 2 units remaining; 2 units or $\frac{16}{8}$ and $\frac{5}{8}$ are $\frac{21}{8}$; $\frac{21}{8} \div 8 = \frac{21}{64}$ (multiplying the denominator); write $\frac{21}{64}$ in fractional column.

Working explanation: 11, 1; 34, 4; $2\frac{5}{8}$ or $\frac{17}{8}$, $\frac{17}{64}$.

(2) Divide $112\frac{3}{4}$ by $7\frac{1}{3}$. Since multiplying both dividend and divisor by the same number does not change the value of the quotient, we may multiply both terms by the L.C.D. of the fractions, in which case we shall obtain integers for the new dividend and divisor. The work appears as below:

$$\begin{array}{r}
 7\frac{1}{3} \overline{) 112\frac{3}{4}} \\
 \underline{12 \quad 12} \\
 88 \quad 1353 \\
 \\
 15\frac{3}{8} \quad \text{Ans.} \\
 88 \overline{) 1353} \\
 \underline{88} \\
 473 \\
 \underline{440} \\
 33 \\
 \underline{88}
 \end{array}$$

Instead of finding the complete quotient in (2) we are sometimes interested in finding the *quotient with remainder*. Thus, we may wish to find how many lengths each containing $7\frac{1}{3}$ yd. can be cut from $112\frac{3}{4}$ yd. and how much will remain. There are three ways of proceeding:

1. Dividing as in (c), we obtain $15\frac{3}{8}$; this means that in $112\frac{3}{4}$ there are 15 times $7\frac{1}{3}$ and $\frac{3}{8}$ of $7\frac{1}{3}$ *more*: that is,

$\frac{3}{8}$ of $\frac{22}{3}$ or $2\frac{3}{4}$ remains. Hence the answer is 15, with $2\frac{3}{4}$

remaining. If measuring pieces of cloth each containing $7\frac{1}{3}$ yards, we could have 15 such lengths with $2\frac{3}{4}$ yards remaining.

2. Proceeding as above, up to the point of finding the integral quotient, we obtain 15; *i.e.*, $7\frac{1}{3}$ is contained in $112\frac{3}{4}$, 15 times; 15 times $7\frac{1}{3} = 110$; subtracting 110 from $112\frac{3}{4}$ we have $2\frac{3}{4}$ remaining.

3. Multiplying both dividend and divisor by the same number does not change the value of the quotient, but multiplies the remainder by that number. In (2) we multiplied both terms by 12; this does not affect the quotient, 15, but has multiplied the remainder; hence the remainder, 33, is twelve times the true remainder. Dividing 33 by 12 we obtain $2\frac{3}{4}$. *Ans.*, quotient, 15, with remainder, $2\frac{3}{4}$.

Mental computations.—In adding or subtracting unit fractions some teachers prefer the device of adding or subtracting the denominators for the numerator of the answer, and multiplying the denominators for the denominator of the answer. Thus:

$$\frac{1}{6} + \frac{1}{4} = \frac{6 + 4}{6 \times 4} = \frac{10}{24} = \frac{5}{12}.$$

$$\frac{1}{4} - \frac{1}{6} = \frac{6 - 4}{4 \times 6} = \frac{2}{24} = \frac{1}{12}.$$

For other fractions multiply the numerator of the first by the denominator of the second, then multiply the numerator of the second by the denominator of the first

and add, or subtract these two results for the numerator of the answer; multiply the denominators for the denominator. For example, add $\frac{5}{6}$ and $\frac{3}{4}$:

$$\frac{20 + 18}{6 \times 4} = \frac{38}{24} = \frac{19}{12} \text{ or } 1\frac{7}{12}.$$

The authors do not advocate this as the best method of procedure, but prefer to proceed as in written work, *having the pupils declare aloud each step in the procedure*; that is, (1) call off the least common denominator, which may easily be found by the method of inspection (p. 144); (2) call off the numerator of each new fraction found by reducing the original fraction to higher terms; (3) call off the answer. Thus in adding $\frac{5}{6}$ and $\frac{3}{4}$, the pupil says 12; 10, 9; $\frac{19}{12}$ or $1\frac{7}{12}$. This method is preferable for several reasons: (1) Unless there is a decided gain in a change of procedure, it is helpful to use the same method for oral and written work. In cases where the steps taken in written work are difficult to hold in mind, it is much better to use a distinct method for oral work, but a comparison of the above methods in adding $\frac{5}{6}$ and $\frac{3}{4}$ will reveal the fact that the latter procedure is no more difficult than the former. (2) By the former procedure a different method is used for unit fractions than for other fractions, which adds one more thing to hold in mind. (3) The denominator thus found is often not the L.C.D., which always means an added step in reducing the result to lowest terms.

The working explanation for a subtraction example is similar to that for addition. Thus in $\frac{5}{6} - \frac{3}{4}$, the pupil says: 12; 10, 9; $\frac{1}{12}$.

In multiplication, proceed as in written work, except that it is easier to omit any possible cancellation, and reduce the result to lowest terms if necessary. Thus, in $\frac{5}{8} \times \frac{3}{4}$, the pupil says $\frac{15}{32}$ or $\frac{5}{8}$. For mental multiplication of a fraction by an integer divide the denominator if possible; if not, multiply the numerator. See page 167 (b).

In division, two methods are possible: (1) Proceed as in written work; invert the divisor and multiply, avoiding cancellation. Thus, in $\frac{5}{6} \div \frac{3}{4}$, the pupil says: $\frac{5}{6} \times \frac{4}{3}$, $\frac{20}{18}$, $\frac{10}{9}$ or $1\frac{1}{9}$. (2) A second method is to reduce the fractions to equivalent fractions having the same denominator (L.C.D.) and then divide the numerators of the resulting fractions. The reason for this procedure may be easily understood by referring back to the second deductive proof of the rule for dividing a fraction by a fraction (p. 172). This method uses the same steps as are employed in adding and subtracting fractions, and is shorter than the other method. Thus in $\frac{5}{6} \div \frac{3}{4}$, the pupil first calls off the L.C.D. 12; then each numerator, 10, 9; then he divides these numerators and calls off $\frac{10}{9}$ or $1\frac{1}{9}$. For mental division of a fraction by an integer, divide the numerator if possible; if not, multiply the denominator (p. 171).

Fractional cases, or type problems. — In the multiplication of an integer by a fraction some interesting problems arise. Let us consider the following statement.

If in a class of 52 pupils $\frac{3}{4}$ of the class are present, there are 39 pupils present. In this statement three terms are involved: $\frac{3}{4}$, 52, and 39. Any one of the terms can be found if the other two are given. Hence the following questions arise:

A. In a class of 52 pupils, $\frac{3}{4}$ of the class is present. How many pupils are present? We must find $\frac{3}{4}$ of 52.

B. If 39 pupils are present in a class of 52, what part of the class is present? Here we must find what part 39 is of 52.

C. If 39 pupils are present and this is $\frac{3}{4}$ of the class, how many are in the class? Our question here becomes: 39 is $\frac{3}{4}$ of what number?

We may increase or decrease a number by a part of itself. Thus, $52 + \frac{3}{4}$ of 52 = 91. $52 - \frac{3}{4}$ of 52 = 13.

In each of these statements three numbers are involved. As before, any one of these can be found if the other two are given. It is customary, however, to consider only the cases which arise from the omission of 52, as:

D. What number increased by $\frac{3}{4}$ of itself gives 91?

E. What number decreased by $\frac{3}{4}$ of itself gives 13?

These so-called *five fractional cases* may be expressed in general terms as follows:

A is called Case I. — To find a fractional part of a number.

B is called Case II. — To find what part one number is of another.

C is called Case III. — To find a number when a fractional part of it is given.

D is called Case IV. — To find what number increased by a fractional part of itself equals a given result.

E is called Case V. — To find what number decreased by a fractional part of itself equals a given result.

Solutions of problems in the five cases. — These may be solved as any other complex problem — by stating for each its component simple problems, and giving their answers (p. 276).

Case I. What is $\frac{1}{4}$ of 52? *Ans.*, 13. If $\frac{1}{4}$ of 52 is 13, what is $\frac{3}{4}$ of 52? *Ans.*, 39. *Working explanation:* 13, 39.

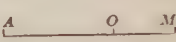
Case II. 1 is what part of 52? *Ans.*, $\frac{1}{52}$. If 1 is $\frac{1}{52}$ of 52, what part of it is 39? *Ans.*, $\frac{39}{52}$. *Working explanation:* $\frac{39}{52}$, $\frac{3}{4}$.

Case III. If 39 is $\frac{3}{4}$ of a number, what is $\frac{1}{4}$ of the number? *Ans.*, 13. If 13 is $\frac{1}{4}$ of a number, what is the number? *Ans.*, 52. *Working explanation:* 13, 52.

Case IV. What number increased by $\frac{5}{8}$ of itself equals 44? If a number is increased by $\frac{5}{8}$ of itself, what is the result? $\frac{13}{8}$ of the number. If $\frac{13}{8}$ of a number is 44, what is $\frac{1}{8}$ of it? 4. If $\frac{1}{8}$ of a number is 4, what is the number? 24. *Working explanation:* $\frac{13}{8}$, 4, 24.

Case V. What number decreased by $\frac{3}{8}$ of itself equals 30? If a number is decreased by $\frac{3}{8}$ of itself, what is the result? $\frac{5}{8}$ of the number. If $\frac{5}{8}$ of a number is 30, what is $\frac{1}{8}$ of it? 6. If $\frac{1}{8}$ of a number is 6, what is the number? 48. *Working explanation:* $\frac{5}{8}$, 6, 48.

EXERCISES

1. To express what fractional part AO is of AM  explain what is necessary.

2. What is the meaning of *denominator*? Define it. Of *numerator*? Define it.



FIGURE 28

3. In the following, tell what is the unit of measure for the fractional part considered: (a) the shaded part of the rectangle; (b) the heavy part of the line;

(c) A sold 49 acres of his field of 240 acres; what part did he sell? (d) 3 students are absent from a class of 29; what part of the class is absent?

4. Is it easier to find a fractional part of a *unit* or a fractional part of a *group*? Tell why you think so.

5. Read the following and give the meaning of each:

$$\frac{5}{9}, \frac{a}{b}, \frac{44}{47}, \frac{31}{32}, \frac{a}{\frac{b}{c}}, \frac{4}{\frac{5}{8}}, \frac{\frac{x}{y}}{\frac{y}{z}}, \frac{5\frac{1}{4}}{7}, \frac{43}{5}, \frac{\frac{5}{8}}{\frac{9}{7}}$$

6. Name all the different kinds of fractions you know, and define each.

7. Add: $15\frac{3}{8}$, $26\frac{7}{9}$, $22\frac{1}{12}$, $16\frac{5}{6}$, and explain how to find the L.C.D. by inspection.

8. Add: $27\frac{5}{112}$, $18\frac{3}{162}$, $24\frac{4}{133}$, and explain how to find the L.C.D. by factoring.

9. Add: $5\frac{4}{11}$, $3\frac{2}{9}$, $4\frac{5}{12}$.

10. Subtract $111\frac{8}{9}$ from $324\frac{7}{12}$, explaining by the Austrian method.

11. From $4\frac{1}{8}$ take $1\frac{9}{7}$.

12. Take $54\frac{3}{109}$ from $83\frac{9}{77}$.

13. Add, subtract, multiply, divide: $6\frac{5}{12}$ and $3\frac{3}{4}$.

14. State the meaning of $\frac{5}{8}$ multiplied by 4; $\frac{5}{8}$ times 4; $\frac{5}{8}$ of 4; 4 multiplied by $\frac{5}{8}$; 4 times $\frac{5}{8}$; $\frac{5}{8}$ multiplied by $\frac{2}{3}$; $\frac{5}{8}$ times $\frac{2}{3}$.

15. Multiply the following: $\frac{5}{8}$ by 2; by 3; by 4; by 6.

16. State the two principles to be used in answering Exercise 15.

17. Multiply by the general rule, stating it: (a) $2\frac{5}{11}$ by $2\frac{1}{3}$; (b) $\frac{5}{16}$ by $\frac{1}{12}$; (c) $76\frac{3}{4}$ by 2; (d) $763\frac{4}{5}$ by 9; (e) $5\frac{9}{11}$ by $2\frac{5}{14}$; (f) $862\frac{3}{8}$ by $71\frac{5}{8}$.

18. Multiply without reducing to improper fractions (c), (d), (e), and (f) in Exercise 17.

19. Divide mentally: $\frac{9}{15}$ by 2; by 3; by 4; by 5; by 6.

20. State the two principles to be used in answering Exercise 19.

21. Divide by the general rule: (a) $2864\frac{3}{4}$ by 7; (b) $58291\frac{3}{7}$ by 8; (c) $42837\frac{3}{4}$ by 8; (d) $723\frac{4}{5}$ by $81\frac{2}{3}$; (e) $\frac{1}{16}$ by $\frac{25}{32}$; (f) $3287\frac{3}{4}$ by 35; (g) $413\frac{5}{8}$ by $18\frac{5}{8}$; (h) $7084\frac{3}{8}$ by $7\frac{1}{2}$.

22. Divide all except (e) in Exercise 21 without reducing the mixed numbers to improper fractions.

23. Express answers as quotients with remainders for (a), (f), and (h) in Exercise 21, making up a practical problem for each.

24. State the six cardinal principles of fractions.

25. Illustrate each of the six principles by an exercise calling for the application of the principle.

26. Find the answers mentally, giving working explanation: (a) $\frac{3}{4} + \frac{2}{5}$; (b) $\frac{5}{6} + \frac{1}{3}$; (c) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$; (d) $\frac{5}{6} - \frac{3}{4}$; (e) $\frac{7}{8} - \frac{2}{3}$; (f) $\frac{5}{6} - \frac{3}{4}$; (g) $\frac{5}{6} - \frac{2}{3}$.

27. Find the answers mentally for the following: (a) $\frac{2}{3} \times \frac{4}{7}$; (b) $\frac{3}{4} \times \frac{5}{6}$; (c) $\frac{1}{2} \times \frac{3}{4}$; (d) $2\frac{1}{2} \times 4$; (e) $6 \times 3\frac{1}{2}$.

28. For mental division reduce the fractions to equivalent fractions having the L.C.D. and divide the numerators: (a) $\frac{3}{4} \div \frac{2}{3}$; (b) $\frac{5}{6} \div \frac{5}{6}$; (c) $\frac{4}{5} \div \frac{2}{3}$; (d) $\frac{7}{8} \div \frac{5}{6}$; (e) $\frac{7}{8} \div \frac{3}{4}$.

29. Write the answers only, finding results mentally:

$$\begin{array}{llll} (a) \frac{5}{6} \times \frac{3}{8} & (c) \frac{3}{11} \times \frac{2}{3} & (e) 1\frac{3}{4} \times \frac{5}{7} & (g) 2\frac{1}{2} \times 3\frac{1}{3} \\ (b) \frac{5}{7} \times \frac{1\frac{1}{2}}{1\frac{1}{2}} & (d) \frac{5}{6} \times \frac{7}{8} & (f) 1\frac{5}{6} \times \frac{9}{11} & (h) 4\frac{1}{2} \times 1\frac{2}{3} \end{array}$$

30. Find the average attendance of two grades if 41 out of 56 are present in one grade, and 67 out of 70 in another. Find the fractional part of each grade and then average the two fractions.

31. Which had the larger fractional part of the class absent and by what fraction: a class of 39 having 4 absent, or a class of 52 having 5 absent?

32. Using mental computations, find the value of:

$$\begin{array}{l} (a) 2\frac{1}{2} + 1\frac{1}{3} - \frac{5}{6} \times 1\frac{1}{2} + \frac{5}{6} \div 1\frac{2}{3} \\ (b) \frac{3}{4} \times \frac{2}{3} + \frac{7}{8} \div \frac{3}{4} - 1\frac{1}{2} \\ (c) \frac{2}{3} + \frac{3}{4} + \frac{5}{6} - \frac{2}{3} \text{ of } \frac{3}{4} \text{ of } \frac{5}{6} \\ (d) \frac{\frac{3}{4}}{1\frac{1}{2}} + \frac{\frac{3}{4} \times \frac{5}{6} \times \frac{9}{10}}{\frac{3}{4}} \end{array}$$

33. Fractional cases. Using mental computations, fill in the blanks:

$$\begin{array}{ll} (a) 16 = \frac{5}{6} \text{ of } — & (f) 42 = — \text{ of } 84 \\ (b) 63 = — \text{ of } 72 & (g) 84 = 1\frac{2}{3} \text{ of } — \\ (c) — = \frac{7}{8} \text{ of } 91 & (h) 34 = — \text{ of } 51 \\ (d) 64 = — \text{ of } 112 & (i) — = \frac{5}{17} \text{ of } 68 \\ (e) 27 = \frac{9}{11} \text{ of } — & (j) 72 = 1\frac{2}{3} \text{ of } — \end{array}$$

(k) $\text{---} = \frac{2}{3}$ of 168

(p) $\text{---} = \frac{1}{16}$ of 57

(l) $18 = \text{---}$ of 117

(q) $104 = \frac{8}{9}$ of ---

(m) $54 = \frac{6}{13}$ of ---

(r) $60 = \frac{3}{7}$ of ---

(n) $52 = \text{---}$ of 65

(s) $96 = \text{---}$ of 104

(o) $\text{---} = \frac{5}{9}$ of 135

(t) $\text{---} = \frac{7}{12}$ of 96

34. Using written computations, fill in the blanks:

(a) $234 = \frac{3}{7}$ of ---

(b) $\text{---} = \frac{3}{180}$ of 5056

(c) $1860 = \text{---}$ of 1829

(d) $\text{---} = \frac{5}{7}$ of 2796

(e) $118 = \text{---}$ of 649

(f) $\text{---} = \frac{1}{100}$ of 3178

(g) $187 = \text{---}$ of 255

(h) $\text{---} = \frac{4}{11}$ of 6885

(i) $68 = \frac{1}{4}$ of ---

35. Prove deductively that multiplying both dividend and divisor by the same number multiplies the remainder by that number.

36. Simplify each, showing by your work the method employed.

(a) $\frac{\frac{2}{3}}{\frac{3}{4}}$

(c) $\frac{\frac{1}{2} + \frac{3}{4}}{\frac{3}{4} - \frac{3}{4}}$

(e) $\frac{2}{3}$ of $4\frac{1}{2}$ of $\frac{3}{7}$ of $4\frac{2}{3}$

(b) $\frac{5\frac{1}{2}}{1\frac{5}{8}}$

(d) $\frac{\frac{3}{4} \times \frac{5}{8}}{\frac{1}{2} - \frac{1}{3}}$

(f) $\frac{\frac{3}{4} \text{ of } \frac{5}{9} \text{ of } \frac{5}{8}}{1\frac{1}{2} \text{ of } \frac{4}{9} \text{ of } \frac{5}{7}}$

37. How many coats, each requiring $4\frac{3}{8}$ yd., can be made from 265 yd. of goods? How many yards remain unused?

38. (a) Show on a yardstick that $\frac{3}{4}$ of 1 ft. = $\frac{1}{4}$ of 3 ft., or that $3 \text{ ft.} \div 4 = \frac{3}{4} \text{ ft.}$ (b) Show on a foot rule that $\frac{3}{8}$ of 1 inch = 3 inches $\div 8$. Draw a generalization.

39. Find the cost of $5\frac{3}{8}$ yd. at \$2.75 per yard.

40. Find the cost of 6 lb. 14 oz. beef at 48¢ a pound.

41. Find the cost of 2 lb. 10 oz. cheese at 35¢ a pound.

42. Explain as to a class: (a) reduction of $\frac{17}{8}$ to a mixed number; (b) reduction of $5\frac{3}{8}$ to an improper fraction; (c) reduction of $\frac{3}{4}$ to 24ths.

43. The density of copper is 550 lb. per cubic foot. How many

feet to a pound for copper wire of diameter (a) $\frac{3}{32}$ "', (b) $\frac{1}{16}$ "', (c) $\frac{9}{64}$ "', (d) $\frac{7}{320}$ "'? The formula for volume of a cylinder is $V = a \times h$, where a represents the number of square units in a cross section and h the number of linear units in the length.

44. In each of the following statements three numbers are involved: $52 + \frac{3}{4}$ of $52 = 91$, and $52 - \frac{3}{4}$ of $52 = 13$; in the first, 52, $\frac{3}{4}$, and 91; in the second, 52, $\frac{3}{4}$, and 13. Any one of these can be found if the other two are given. State the six questions that arise from these statements by the omission of a term. Which of these six types are not included in the five fractional cases?

METHODS OF TEACHING

Sequence of topics as determined by child's needs. — A child's knowledge of fractions begins very early, with his use of the expressions *half* and *quarter*. These terms are not at first well defined; thus, the child is wont to speak of the *biggest half*, meaning the larger *portion*. The main purpose of the work with fractions in the early grades is that of *definition*, to give the child a clear working notion of a fraction and of the meaning and use of the numerator and of the denominator. By this we do not mean that the child should learn any formal definitions of these terms; he comes to understand them through his experience with concrete objects and through the solution of problems. The formal definitions will come later.

Children have occasion to use halves, fourths, and thirds in connection with measuring and construction work; in the third year (usually) they will learn the partition facts, which involve the use of the unit fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . $\frac{1}{9}$; and a little later they will be solving problems like, What is the cost of $\frac{3}{4}$ of a pound of cheese at 36 cents a pound?

In most schools the systematic study of fractions comes in the fifth year, though some schools teach addition and subtraction late in the fourth year, and some defer division until the sixth year.

Early concrete work. — This work involves physical activities, observation, and oral expression by the children; the material used should be varied and as intrinsically interesting as possible, though much of it cannot be very closely identified with the child's needs at the time. Squares, rectangles, circles, and lines drawn on the blackboard or on paper are separated by means of lines into the parts to be studied; strips of paper of uniform length are folded and then shaded or colored; objects in the room, the desks, books, sticks, the children themselves, are used in studying parts of groups; the blackboard is separated into equal parts in preparation for class work at the board; papers are folded into fourths, sixths, or eighths, in preparation for seat work.

Some of these experiences just happen, so far as the children are aware; but *merely* incidental uses of fractional parts are not enough; the children must not only have such experiences but these experiences must be so employed by the teacher as to engage the child's conscious attention¹ to the fractional relations involved. The objects and activities are but the raw materials from which ideas — concepts — are to be constructed. There are many ways of securing this conscious attention; probably the most common way is through the child's oral response to questions, but this is by no means the only way. The

¹ BAGLEY, W. C. — *The Educative Process*, pp. 146-148; The Macmillan Company.

following lesson illustrates how the teacher may develop a situation, or a series of situations, for the purpose of developing ideas about *fourths*. It is assumed that the children are familiar with halves. Italics are used to indicate the teacher's part in the discussion.

I would like eight children to go to the front board. The board is marked off into four equal parts. Will you divide up so that there will be the same number of children at each part of the board? After the children are properly placed, there will be questions and discussion leading to the following conclusions:

1. We divided the board into four equal parts. Each part is called one fourth of the board.

2. We divided 8 children into four equal groups. Each group is called one fourth of the children. One fourth of 8 children are 2 children.

The children at the left half of the board will make a square one foot on each side. Those at the right half of the board will each draw a circle. This done, bring out the following:

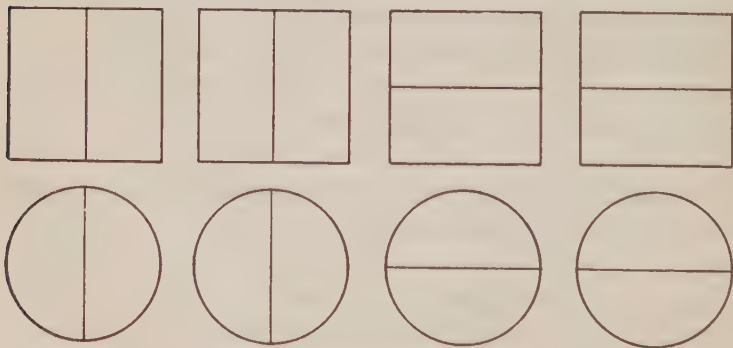


FIGURE 29

3. Two fourths of the board are in the left half and two fourths in the right half.

4. One half of the children are two fourths of them

5. One half of the pictures are squares, the other half are circles; two fourths of the pictures are squares, and two fourths of them are circles.

6. One half of 8 pictures is 4 pictures. Two fourths of 8 pictures are 4 pictures.

The children at the first and second parts of the board will draw a vertical line to divide their picture into halves; those at the third and fourth parts of the board will draw a horizontal line to divide their picture into halves.

Now have the first child divide his square into fourths; have the second child divide his square into fourths a different way, etc., with the results shown below.



FIGURE 30

Proceed similarly with the circles. If a child attempts to use parallel lines, show him that he cannot be sure the parts are equal. Number the pictures from 1 to 8. (Let 8 other children take up the work.)

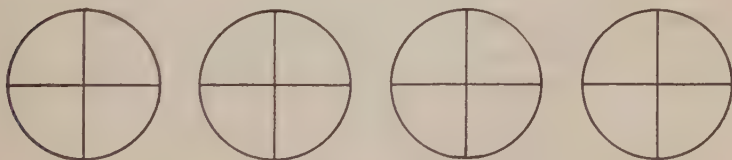


FIGURE 31

With colored crayon shade one fourth of square number 1. Next, shade two fourths of number 2, two fourths of number 3 in a different way, three fourths of number 4. Proceed similarly for the circles.

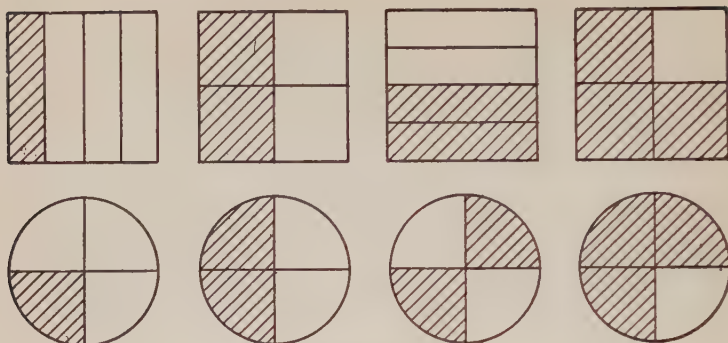


FIGURE 32

Which pictures have one fourth shaded? Which have one fourth plain? One half shaded? Three fourths shaded? Three fourths plain?

The lesson unit described above is too much for a single class period; it could well be separated into two or three parts, saving the board work from day to day until the lesson is completed.

The following relations connect the work in fractions with that in measurements: 1 pint = $\frac{1}{2}$ quart; 1 cupful = $\frac{1}{2}$ pint = $\frac{1}{4}$ quart; expressing $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{5}{6}$ of a foot in inches; expressing $\frac{1}{3}$ and $\frac{2}{3}$ of a yard in feet.

The six fundamental principles. — The concrete work suggested in the previous section prepares the way for a conscious study of the principles of fractions. These principles are not to be introduced all at one time and studied for their own sake, but are taken up when needed in working examples. This means that the order on page 163 is not the teaching order. The first principle needed is number three (Multiplying both terms by the same number . . .), used in reducing fractions to a

common denominator preparatory to addition. The second principle to be used is number six (Dividing both terms . . .), which finds application in reducing to lowest terms. The other principles occur in connection with multiplying and dividing fractions, as we shall see later.

This change in the order of the principles makes it impossible to teach them by the deductive method, and the immaturity of the pupils furnishes additional reason for using the inductive method in the early work. The proper time for the deductive treatment of principles is in the higher grades after the pupils have a sufficient background to understand the reasons upon which the development is based, or after the work in fractions has been completed, and a review is in order. Such a review might well start with a lesson on the six principles, and continue with a series of lessons each having for its aim a study of a single principle, and of the various ways the principle helps one in working with fractions.¹

Inductive development of principles. — There is available a great variety of objective aids for use in teaching the principles of fractions. On page 160, lines were employed. There is reason to believe that rectangles are better for children. However, the exact nature of the device is not important so long as it is simple to manipulate, easy on the eyes, and clearly representative of the relation to be studied. In contrast to the great variety of concrete material used in the early grades (p. 183) it is advisable later to use more conventional material.

¹ This is in harmony with the principle that a review of a topic should take it up from a new point of view and with a different organization. In the development of fractions the organization is determined by practical needs; in the review the organization may well be determined by the logic of the subject.

Some teachers would use the same kind of material with every principle, believing that this makes for economy in learning. Figure 33 represents a common type of visual aid in the study of the principles of fractions. The idea can be worked out in a series of charts, or in drawings on the board, or the children may cut and fold strips of paper, as in the diagram :

Unit							
$\frac{1}{2}$				$\frac{1}{2}$			
$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$	
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

FIGURE 33

In Fig. 33 we see that $\frac{1}{2} = \frac{2}{4}$, $\frac{1}{4} = \frac{2}{8}$, $\frac{2}{4} = \frac{4}{8}$, $\frac{3}{4} = \frac{6}{8}$, and $\frac{1}{2} = \frac{4}{8}$. We may conclude that multiplying both terms of a fraction by the same number does not change the value of the fraction. In the same figure we see that $\frac{1}{2}$ is twice as large as $\frac{1}{4}$, or $2 \times \frac{1}{4} = \frac{1}{2}$; $\frac{1}{4}$ is twice as large as $\frac{1}{8}$, or $2 \times \frac{1}{8} = \frac{1}{4}$; $\frac{1}{2}$ is 4 times as large as $\frac{1}{8}$, or $4 \times \frac{1}{8} = \frac{1}{2}$; and $\frac{3}{4}$ is twice as large as $\frac{3}{8}$, or $2 \times \frac{3}{8} = \frac{3}{4}$. We may conclude that dividing the denominator of a fraction multiplies the fraction.

The further use of these figures is left to the reader (see the exercises).

Addition. — The following types are arranged in order of difficulty :

1. Denominators alike, as $\frac{1}{4} + \frac{1}{4}$; $\frac{5}{8} + \frac{7}{8}$.
2. Largest denominator the lowest common denominator, as $\frac{1}{2} + \frac{1}{4}$; $\frac{1}{3} + \frac{1}{3}$; $\frac{5}{8} + \frac{3}{4}$.

3. Same as (1) and (2), except that there are three or more addends arranged in column.

(3)

$$\begin{array}{r} \frac{3}{8} \\ \frac{1}{8} \\ \frac{5}{8} \end{array}$$

$$\begin{array}{r} \frac{5}{8} \\ \frac{3}{8} \end{array}$$

4. Column addition, the lowest common denominator larger than any given denominator.

$$\begin{array}{r} \frac{5}{8} \\ \frac{1}{2} \end{array} \quad (4)$$

$$\begin{array}{r} \frac{2}{3} \\ \frac{1}{4} \end{array} \quad \begin{array}{r} \frac{3}{4} \\ \frac{1}{8} \end{array} \quad \begin{array}{r} \frac{2}{3} \\ \frac{3}{4} \\ \frac{1}{8} \end{array}$$

5. Mixed numbers, no carrying from fractions to units.

$$\begin{array}{r} 5\frac{3}{8} \\ 46\frac{1}{4} \end{array}$$

$$34\frac{1}{8}$$

6. Mixed numbers with carrying.

$$\begin{array}{r} 5\frac{3}{8} \\ 46\frac{1}{4} \end{array}$$

$$26$$

$$15\frac{5}{8}$$

(a)	(b)	(c)	(d)
		$13\frac{5}{8}$	$4\frac{7}{16}$
$6\frac{5}{8}$	$5\frac{5}{8}$	$7\frac{7}{12}$	$6\frac{11}{10}$
<u>$4\frac{7}{8}$</u>	<u>$7\frac{11}{12}$</u>	<u>$13\frac{5}{8}$</u>	<u>$4\frac{7}{12}$</u>

In the early work with types 1 and 2 the pupil should get the answers through the use of objects. From materials similar to Figure 33 such examples as these could be worked: $\frac{1}{3} + \frac{1}{3}$, $\frac{3}{6} + \frac{2}{6}$, $\frac{2}{3} + \frac{1}{6}$, $\frac{5}{6} + \frac{1}{12}$. For finding sums exceeding 1 the foot rule is better, because the eye and the thought can pass without difficulty into the second inch in obtaining the answers to $\frac{3}{4} + \frac{3}{4}$, $\frac{1}{2} + \frac{3}{4}$, and the like; these results could be expressed as improper fractions and as mixed numbers, giving a clear notion of the reduction involved.

When the pupils are fairly proficient in the objective work, they may work the same kind of examples in abstract form for the purpose of becoming accustomed to the algorithm before taking up harder examples.

$$\begin{array}{r} \frac{2}{3} \\ \frac{2}{3} \\ \hline 1\frac{1}{3} \end{array} \quad \begin{array}{r} 2 \\ 2 \\ \hline 4 \end{array}$$

The things to be emphasized in teaching types 3 and 4 are the mechanical mastery of the algorithm (p. 165),

finding by inspection the lowest common denominator (p. 144), the reduction of the addends to the common denominator, and the meaning and process of reducing an improper fraction to a whole or mixed number. While the final objective is the mastery of the abstract processes and their synthesis into the total procedure, it will probably be necessary at times to refer to previous objective experience; otherwise there is danger that the pupils will lose their sense of the meaning of what they are doing. It is unwise to dwell too long upon type 3 before taking up type 4; there is danger that the pupil will over-learn the procedure and get a fixed idea that the largest denominator is always the common denominator.

A few examples of type 5 will be sufficient to accustom the children to the slight change in algorism required in adding mixed numbers; this is just a matter of mechanics.

With type 6 it is well to begin with a few examples in which the denominators are alike, so that the matter of carrying from fractions to units shall not be complicated by the presence of other difficulties. The explanation of carrying from units' column to tens' column in example (b) is a good preparation for understanding carrying from eighths' column to units' column in example (a). The analogy

(a)	(b)
$4\frac{5}{8}$	45
<u>$6\frac{7}{8}$</u>	<u>67</u>

is made quite close by using the *same figures*. In (b) we have units' column and tens' column, in (a) we have eighths' column and units' column; in (b) we add 5 and 7, getting 12 *units* (1 ten and 2 units); in (a) we add 5 eighths and 7 eighths, getting 12 *eighths* (1 unit and 4 eighths); in (b) we write 2 *units* and carry 1 to the tens' column, in (a) we write 4 *eighths* and carry 1 to the units' column.

Subtraction. — As in addition so in subtraction, the teacher should have in mind the different types of examples that arise in order that he may present them in order of progressive difficulty. Such classification, of course, is not to be given to the pupils, but is helpful to a teacher in sensing the difficulties that arise. In subtraction of fractions the terms may be simple fractions, or one or both terms may be mixed numbers; the denominators may be either alike or unlike, and, if unlike, one denominator may or may not be the common denominator; also the fraction in the minuend may or may not be less than the fraction in the subtrahend. These sets of alternatives, taken singly and in combination, yield the following classification:

I. Simple fractions.

(a) Denominators alike.

(b) Denominators unlike.

1. The larger denominator the common denominator.

2. The larger denominator not the common denominator.

II. One, or both, terms mixed numbers.

(a) No carrying (or borrowing).

1. No fraction in the subtrahend.

2. Fractions in both terms.

(b) Carrying (or borrowing).

1. No fraction in minuend.

2. Fractions in both terms.

It should be perfectly obvious that whichever method of subtraction (p. 54) a child uses with integers he should continue to use with mixed numbers. If in Example (a) he thinks: *5 and 6 are 11 (carry 1), 3 and 2 are 5*: then in (b) he should learn to think: *5 sixths and 2 sixths are 7 sixths ($1\frac{1}{6}$), 3*

(a)	(b)
51	$5\frac{1}{6}$
25	$2\frac{5}{6}$

and 2 are 5. If in (a) he thinks: 5 from 11 are 6, 2 from 4 are 2: he must likewise say in (b) 5 sixths from 7 sixths are 2 sixths, 2 from 4 are 2. If he has been thinking: 5 from 11 are 6, 3 from 5 are 2: he should continue in the same thought, saying: 5 sixths from 7 sixths are 2 sixths, 3 from 5 are 2.

But at the time children are taking up subtraction of mixed numbers the subtraction of integers is an old story — the need for oral practice on the working explanation has long since passed — and it is probable, or at least possible, that the teacher herself doesn't know what method, or methods, the children habitually use. It is therefore necessary to find out how the children are subtracting; and if they are not all using the same method, it will be advisable to separate them into groups and to teach each group subtraction of fractions by its own method. This grouping must be adhered to until the habit proper to each group is well launched and the children are ready to carry on individual practice work. Fortunately, the same list of examples, taken in the same order, will serve for whatever method is used.

A few suggestions as to the teaching of the types under II (b) are in order. Where there is no fraction

(a)	(b)
9	90
$3\frac{2}{5}$	<u>32</u>
(c)	(d)
$9\frac{2}{5}$	92
<u>$3\frac{4}{5}$</u>	<u>34</u>

in the minuend, as in (a), the example is analogous to one in subtraction of whole numbers in which 0 stands in units of the minuend as in (b). As this is a particular case, the method of presentation is similar to that for Example (c).

It is well at first to avoid examples in which the denominators are different. The analogy between examples

(c) and (d) is similar to that between examples (a) and (b) in addition of mixed numbers. In (d) we have units' and tens' columns, in (c) fifths' and units' columns; in (d) we increase 4 units by 8 units to get a sum ending in 2 units (1 ten and 2 units), in (c) we increase 4 fifths by 3 fifths to get a sum ending in 2 fifths (1 unit and 2 fifths); in (d) we carry 1 ten to tens' column and say *4 and 5 are 9*, in (c) we carry 1 unit to units' column and say *4 and 5 are 9*.

First Italian method. In (d) 9 tens and 2 units are thought of as 8 tens and 12 units; in (c) 9 units and 2 fifths are thought of as 8 units and 7 fifths.

Second Italian method. In (d) 10 units are added to the minuend and their equivalent 1 ten is added to the subtrahend; in (c) 5 fifths are added to the minuend and their equivalent 1 unit is added to the subtrahend.

Until the habit, whatever it is, is well established it will be necessary to give daily practice on the working explanation.

Multiplication. -- *Types of examples.* -- Textbooks differ somewhat in their organization of this topic. Most of the older textbooks and a few modern ones over-emphasize examples like $\frac{3}{4} \times \frac{5}{8}$, $\frac{2}{3} \times \frac{2}{5} \times \frac{6}{7}$, etc. Such examples must, of course, be included, but as compared with the mental multiplication of $\frac{5}{8}$ by 4, and finding a fractional part of a whole number, they are of little practical use.

The organization given here fairly represents the best modern tendency in that it recognizes considerations of utility as highly important in determining the order in which, and the method by which, the various types of examples should be taught.

- I. Finding a fractional part of a whole number.
 a. Mental-oral. *E.g.*, $\frac{3}{4}$ of 24, $\frac{7}{8}$ of 40, etc.
 b. Written. *E.g.*, $\frac{5}{8}$ of 52, $\frac{3}{4}$ of 632, etc.
- II. Mental-oral multiplication of a fraction by a whole number.
 a. By dividing the denominator. *E.g.*, $2 \times \frac{3}{8}$, etc.
 b. By multiplying the numerator. *E.g.*, $2 \times \frac{3}{7}$, etc.
- III. Written multiplication of a whole number by a mixed number.

$$\begin{array}{r} E.g., 378 \quad 378 \\ \underline{6\frac{3}{4}} \quad \underline{36\frac{3}{4}}, \text{ etc.} \end{array}$$

- IV. Written multiplication of a mixed number by a whole number.

$$\begin{array}{r} E.g., 47\frac{3}{4} \quad 47\frac{3}{4} \\ \underline{6} \quad \underline{26}, \text{ etc.} \end{array}$$

- V. Written multiplication of two or more simple fractions, or small mixed numbers. *E.g.*, $\frac{3}{4}$ of $\frac{5}{8}$, $\frac{2}{3} \times \frac{2}{3} \times \frac{3}{4}$, $3\frac{1}{2} \times 2\frac{3}{4}$, etc.
- VI. Written multiplication of two large mixed numbers. *E.g.*, $473\frac{3}{4}$
 $\underline{56\frac{1}{2}}$

It is very easy to find practical applications of the examples under I and II, less easy to find them for the examples under III and IV, rather difficult in the case of V, and very difficult in the case of VI. A generation ago it was a common practice to teach type V by the rule, and then to apply this rule to all other types of examples. Thus, to find $\frac{3}{4}$ of 24, the child would write

$$\begin{array}{c} 6 \\ \frac{3}{4} \times \frac{24}{1} = 18; \end{array}$$

to find $2 \times \frac{3}{8}$ he would write

$$\begin{array}{c} \frac{2}{1} \times \frac{3}{8} = \frac{3}{4}; \\ 4 \end{array}$$

and to work the examples under III, IV, and VI the child would have to reduce the mixed numbers to improper fractions and then arrange the work as in the other illustrations. Now, if types I and II were of slight practical value, the practice just described would be unquestionably the correct one; the fact that the method for V is cumbersome for I and II would have very little weight. But in view of the frequent occurrence of I and II, it would be a pity to handle them by any but the best methods.

Teaching type I. — Mental examples of this type include finding fractional parts of multiples of the denominator which occur as products in the multiplication tables, and in other known multiplication facts—*e.g.*, $\frac{5}{8}$ of 72 and $\frac{2}{3}$ of 96. Oral practice in such examples will naturally occur in connection with the work in mental multiplication and division. The more important applications occur in connection with finding the cost of fractions of a pound of cheese or meat, or fractions of a yard of cloth or trimming; and in these connections the only denominators used are 2, 4, 8, and (sometimes) 16. However, fractions with other denominators are sometimes used, and must receive attention along with the rest.

In drill work, as well as in the development, advantage should be taken of the connection between the fractional parts and the multiplication and division facts. For example, drill on the combinations in the table of 8 *times* would be followed by drill on the corresponding partition facts, $\frac{1}{8}$ of 8, $\frac{1}{8}$ of 16, etc., and these in turn would lead into finding $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$ of the same numbers.

For the written work under type I see page 167.

Type II. — The working rule for these examples is: Divide the denominator if possible; otherwise, multiply the numerator. The advantage of dividing the denominator where it is possible is seen in $2 \times \frac{3}{8} = \frac{3}{4}$, for by so doing the result in lowest terms is obtained in one step instead of in two. If type IIb is taught before IIa the pupils are less likely to see chances to divide the denominator. The principle, *Dividing the denominator multiplies the fraction*, is here met for the first time; hence it is necessary to give the pupils concrete illustrations of its nature and operation. The objective materials on page 188 will help; also the following:

2 times 1 pint = 1 quart

2 times 3 pints = 3 quarts, etc.

2 times any number of pints = the same number of quarts

2 times 1 eighth = 1 fourth

2 times 3 eighths = 3 fourths, etc.

2 times any number of eighths = the same number of fourths

In every case we have performed the multiplication not by multiplying the *number* of units, but by multiplying their *size*.

When a child has learned to multiply a fraction by dividing its denominator, the other method may be introduced by giving without warning an example like $2 \times \frac{2}{5}$ along with examples of the other type. Most children will do nothing with it; some may get the answer $\frac{2}{2\frac{1}{2}}$; possibly somebody will discover the new method.

The principle, *Multiplying the numerator multiplies the fraction*, may be illustrated objectively (p. 160), or it may be reasoned out thus:

2 times 2 feet are 4 feet

2 times 2 dollars are 4 dollars

2 times 2 marbles are 4 marbles, etc.

2 times 2 of anything are 4 of those things

2 times 2 fifths are 4 fifths, or $2 \times \frac{2}{5} = \frac{4}{5}$

In every case we have multiplied the number of units without changing their size.

The working rule with which we started our discussion may be stated somewhat more concretely as follows: *To multiply a fraction by an integer multiply the size of the parts or multiply the number of parts taken.*

Type III. — The thought steps in mastering this type are very simple. It is possible to introduce the type through a problem, thus: Find the cost of $36\frac{3}{4}$ acres at \$378 an acre. Following the idea used in teaching multiplication by a number of two orders, we first find the cost of $\frac{3}{4}$ acre, then find the cost of 36 acres and add.

The algorism resembles closely the one for multiplication of integers. There are several ways of arranging the work of multiplying by the fraction, three of which are here shown. Some claim that when the form at the left is used, the children are likely to add in the 1134; the other forms avoid this possibility.

$$\begin{array}{r}
 378 \\
 36\frac{3}{4} \\
 4 \overline{)1134} \\
 \underline{283\frac{1}{4}} \\
 2268 \\
 1134 \\
 \underline{13891\frac{1}{4}}
 \end{array}$$

$$\begin{array}{r}
 378 \\
 36\frac{3}{4} \\
 \underline{283\frac{1}{4}} \\
 2268 \\
 1134 \\
 \underline{13891\frac{1}{4}}
 \end{array}$$

$$\begin{array}{r}
 4 \overline{)1134} \\
 \underline{283\frac{1}{4}}
 \end{array}$$

$$\begin{array}{r}
 378 \quad 1134 \\
 36\frac{3}{4} \\
 \underline{283\frac{1}{4}} \\
 2268 \\
 1134 \\
 \underline{13891\frac{1}{4}}
 \end{array}$$

Type IV. — Of the three algorithms given below the first one is clearly the best. It requires fewer combinations, is more like the forms children have used with

$$\begin{array}{r} 47\frac{3}{4} \\ 6 \\ \hline 286\frac{1}{2} \end{array}$$

$$\begin{array}{r} 47\frac{3}{4} \\ 6 \\ \hline 4\frac{3}{4} \\ 282 \\ \hline 286\frac{1}{2} \end{array}$$

$$47\frac{3}{4} = \frac{191}{2} \times \frac{3}{2} = \frac{573}{2} = 286\frac{1}{2}$$

$$\begin{array}{r} 47 \\ 4 \\ \hline 188 \\ 3 \\ \hline 191 \end{array}$$

integers, and is very simply explained. The explanation may use as the apperceptive basis the addition of mixed numbers, the carrying in which is already understood; or the multiplication of $47\frac{3}{4}$ may be associated with the multiplication of 473, the carrying from *fourths* to *units* being analogous to the carrying from *units* to *tens*.

$$\begin{array}{r} 47\frac{3}{4} \\ 47\frac{3}{4} \\ 47\frac{3}{4} \\ 47\frac{3}{4} \\ 47\frac{3}{4} \\ \hline 286\frac{1}{2} \end{array}$$

Type V has been fully treated on page 168. The details of the inductive presentation may safely be left to the reader; probably the rectangular device is easier for children to understand than is the line. It is important that the drawing of the figure be carried out step by step to keep pace with the thought steps of the development; this makes it possible for the learner to visualize one idea at a time. It is desirable, also, that the children should make their own drawings; there is no better way to secure attention to the development.

Type VI, if taught at all, may be deferred until the children learn to multiply algebraic binomials (p. 170). The process may, of course, be related to multiplication of two-order integers each regarded as a binomial made up of tens and units. The process meets no need of ordinary living.

Division. — *Types of examples.* — What was said about the types in multiplication (p. 194) might, with a few changes in words, be repeated in defense of the following organization of division. The types will be seen to be closely analogous to types II, III, IV, V in multiplication.

- I. Mental-oral division of a fraction by an integer.
 - a. By dividing the numerator. *E.g.*, $\frac{8}{9} \div 4$.
 - b. By multiplying the denominator. *E.g.*, $\frac{3}{4} \div 2$.
- II. Written division of a mixed number by an integer. *E.g.*,

$$\begin{array}{r} 7 \overline{)476\frac{2}{3}} \quad 27 \overline{)476\frac{2}{3}} \end{array}$$
- III. Written division by a mixed number.
 - a. Dividend an integer. *E.g.*, $3\frac{1}{2} \overline{)468}$.
 - b. Dividend a mixed number. *E.g.*, $3\frac{1}{2} \overline{)468\frac{2}{3}}$.
- IV. Division, by inverting the divisor, of a fraction by a fraction.
E.g., $\frac{3}{4} \div \frac{2}{3}$, $4\frac{1}{2} \div 2\frac{2}{3} = \frac{9}{2} \div \frac{8}{3}$.

Teaching. — The principles and methods applicable to these types are so fully discussed on pages 171–173 that a more detailed treatment is unnecessary.

Practical uses of common fractions. — In our discussion of the *teaching* of fractions we have used in our examples fractions with small denominators such as are likely to occur in everyday affairs. This is in harmony with the best current practice. Modern textbooks are omitting such examples as the following, taken from a text widely used twenty-five years ago :

Change $\frac{2}{3}$ to 196ths.

Reduce $\frac{8}{11}\frac{5}{8}$ to lowest terms.

Reduce to an improper fraction $72\frac{2}{3}$.

Reduce to a whole or mixed number $\frac{51417}{118}$.

Add $32\frac{1}{3}$, $18\frac{1}{4}$, and $45\frac{2}{3}$.

Find the product of $47 \times 6\frac{5}{13} \times 5\frac{1}{6}$.

Divide $\frac{8}{11} \times \frac{1}{17} \times 5\frac{1}{8}$ by $18\frac{3}{4}$.

Many courses of study are carefully defining the work in fractions so as to exclude useless and time consuming examples. One course, which has been in vogue several years, prescribes that no denominator greater than 64 shall occur, and that no prime number above 11 nor multiple of a prime above 7 shall be used as a denominator.

Any teacher who stops to think about the matter will see at once that the life demands for common fractions are extremely limited, but so strong is the power of tradition that to overcome the inertia of established custom it has been necessary to make careful studies with a view to determining what fractions are in popular use and what fractions are required in important technical situations. Lennes¹ has compiled a list of fractions used in various lines of trade; the stationer uses certain fractions which occur in the standard sizes of paper; there are standard widths of ribbon, standard measurements of screws, nails, bolts, wire, steel plate, shot, bullets, etc. His conclusion is that "the only numbers used as denominators are: 2, 3, 4, 6, 8, 16, 20, 32, 40, 64, 80, 100, 128, 160, 320, 640, 1280, 2560." Wilson² found in 14,583 problems reported by adults as having occurred in their experience, that fractions appeared 1938 times; the denominators were: 2 (1289 times), 4 (348 times), 3 (173 times), 8 (45 times), 10 (27 times), 12 (12 times), 6, 9, 16, 11, 5, 7, 24, 32, 15, 17, 18, 25, 100. These are arranged in order of their frequency of occurrence, the frequency ranging from 8 down to 1 where not specifically stated. It appears that there are only one-seventh as many fractions as there are

¹ LENNES, N. J. — *The Teaching of Arithmetic*, p. 251.

² WILSON, G. M. — "A Survey of the Social and Business Usages of Arithmetic"; *Teachers College Contributions to Education*, No. 100, 1919.

problems, and that only one per cent of the denominators exceed 12.

For teaching purposes we can safely confine ourselves to the use of fractions with small denominators, *provided that the instruction is such as to develop intelligence in the use of fundamental principles.*

EXERCISES

1. In chapter I (p. 17) we discussed how the child first gets a concrete idea of number, then learns to express it verbally, and finally learns to express it in symbols. Are these steps followed in the early work with fractions? Explain.

2. In the early work with fractions which is the better order to follow, *halves, fourths, eighths, thirds, sixths, ninths, fifths, tenths, sevenths*, or *halves, thirds, fourths, fifths, sixths, sevenths, eighths, ninths, tenths*? Is there an order better than either of these? Explain.

3. Work out in detail a lesson on *sixths*, assuming that *halves* and *thirds* have been studied.

4. Tell how one might use Fig. 33 (p. 188) in teaching principles *one, two, and three*.

5. Make a chart similar to Fig. 33 for teaching *thirds, sixths, and twelfths*; for comparing *thirds* and *fourths* by means of *twelfths*.

6. Suppose you wish the children to work out objectively the relations between *halves, thirds, and sixths*. Give exact directions, demonstrating as you go along, for the children to work out the relations through paper folding.

7. In teaching the first lesson on addition of mixed numbers with carrying, (a) What examples would you use? (b) What concrete material would the children use? (c) What thought steps would be taken?

8. Tell how you would teach finding the lowest common denominator in examples of type 4 in addition. See page 189.

9. How would you help children to understand these reductions: (a) mixed number to an improper fraction, (b) a fraction to lowest terms?

10. Prepare an outline for the types of examples on page 189 according to the plan on page 191.

11. Prepare a list of examples illustrating the types in the outline on page 191.

12. How would you go at it to find out what method of subtraction each child in a class of 40 habitually uses?

13. Explain as to a class each type of example under II, page 191 (a) by the Austrian method, (b) by the first Italian method, (c) by the second Italian method.

14. Why is it sometimes helpful to children to have fractions expressed in the form *3 fourths* instead of in the customary form $\frac{3}{4}$? Mention a situation in which you would make this change.

15. Outline the content of a lesson on finding $\frac{1}{6}$, $\frac{2}{6}$, etc. of integers, and describe in detail how you would teach the lesson.

16. Using some other multiplier, as 3 or 4, work out a series of statements about units of measure leading up to (a) 3 times $\frac{2}{3} = \frac{2}{1}$, (b) 3 times $\frac{2}{3} = \frac{2}{1}$, (c) 4 times $\frac{3}{4} = \frac{3}{1}$, (d) 4 times $\frac{3}{4} = \frac{3}{1}$.

17. Tell how you would use Fig. 33, page 188, in teaching the multiplication of a fraction by dividing the denominator. List the examples that could be worked with the device mentioned.

18. Tell how you would use the foot rule with the inches divided into 8ths in teaching (a) multiplication of a fraction by dividing the denominator, (b) multiplication of a fraction by multiplying the numerator. In what respect is the foot rule a better device than the figure on page 188?

19. Tell how you would use the device on page 188 in teaching (a) dividing a fraction by dividing the numerator, (b) dividing a fraction by multiplying the denominator. What examples could the children work with the device chosen?

20. Tell how you would do the same work with the aid of the foot rule.

21. Starting with statements like $2 \text{ yards} \div 3 = 2 \text{ feet}$, develop the division of a fraction by multiplying the denominator.

22. As in Exercise 21, lead up to $\frac{3}{4} \div 4 = \frac{3}{16}$.

23. Show how addition of fractions could be used as the point of departure in teaching examples like $3 \times \frac{2}{3}$.

24. Show how subtraction of fractions could be used as the point of departure in teaching examples like $\frac{9}{7} \div \frac{2}{7}$. What rule does this suggest? Could you use this rule in examples like $\frac{3}{8} \div \frac{1}{4}$? Compare with the inductive treatment of the example $\frac{5}{7} \div \frac{2}{3}$ on page 171.

25. Explain as to a class the carrying in the example: Multiply $36\frac{2}{3}$ by 6.

26. Explain as to a class the last step in dividing $335\frac{2}{3}$ by 4 by short division.

27. In working examples like (a) what principle is used in getting rid of the fraction in the divisor? This is the first occasion the pupils have for using this principle. How could the children make sure by experiment that the principle is true?

$$(a) \quad 3\frac{1}{2} \overline{)468}$$

$$\begin{array}{r} 7 \overline{)936} \\ 133\frac{2}{3} \end{array}$$

28. Explain the two methods illustrated in (b) and (c). Which method shows the higher skill? Which one do you prefer for your own use? Which one would you teach, and why?

$$(b) \quad 3\frac{1}{2} \overline{)468\frac{2}{3}}$$

$$\begin{array}{r} 7 \overline{)937\frac{1}{3}} \\ 133\frac{1}{3} \end{array}$$

29. One method of teaching the rule for dividing a fraction by a fraction (invert the divisor and multiply) consists in working some examples by the rule, and then checking the results by multiplication. Make a detailed plan using this method. Discuss the merits of this plan as compared with the methods on pages 171, 172.

$$(c) \quad 3\frac{1}{2} \overline{)468\frac{2}{3}}$$

$$\begin{array}{r} 133 \\ 21 \overline{)2812} \\ 21 \\ \hline 71 \\ 63 \\ \hline 82 \\ 63 \\ \hline 19 \end{array}$$

30. Prepare a graded list of examples and problems for practice on each type in multiplication.

31. Prepare a similar list for each type in division.

CHAPTER VIII

DECIMALS

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions. — As explained in Chapter I, we employ the decimal system of notation in writing integers. The principle of this system is that ten units of any order make one of the next higher order. Or we may state it thus: A figure in any order expresses one-tenth of what it would express in the next order to the left. It is perfectly logical to extend this principle to the right of units' order. One-tenth of a unit, or a *tenth*, should therefore be written in the next order to the right of units; one-tenth of a hundredth, or a *thousandth*, in the next order to the right of hundredths, and so on. Our orders by this plan would then be as follows, carried 6 orders to the left and 6 orders to the right of units:

7	0	4	5	6	8	9	3	2	4	6	8	9
millions	hundred-thousands	ten-thousands	thousands	hundreds	tens	units	tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths

When the order names are not written, some device must be used to determine units' order. A point may be placed after units. Thus to write 4 units 3 tenths 6 hundredths we may write 4.36. Such a number is called a *decimal*. It may be defined as a number which results from extending the principle of the decimal notation to the right of units.

This number, it is observed, is always a fractional part of a unit. Furthermore, it is a fraction expressed as tenths, or hundredths, or thousandths, etc. — that is, it is a fraction whose denominator is a power of 10. A second definition for a decimal may be given in this form: A decimal is a fraction whose denominator is a power of ten, in which the denominator is not written but is expressed by the position of the numerator.

Reading and writing. — When the underlying principle of expressing decimals is understood, their reading and writing becomes a simple matter. To read a decimal, read the numerator of the decimal as though it were an integer and then call the name of the right-hand order.

Illustration: .0548 is read: *Five hundred forty-eight ten-thousandths*. In reading an integer and a decimal it is customary to use the word *and* after reading the integer.

Thus, 5.07 is read: *Five and seven hundredths*.

To write a decimal, write the numerator of the decimal as though it were an integer and then place the decimal point so that the name of the right-hand order shall be the same as the name of the decimal.

Thus, to write fifty-four hundred-thousandths we write 54; then pointing to 4, we say "hundred-thousandths"; to 5, we say "ten-thousandths"; prefixing a cipher, we say "thousandths"; another cipher, "hundredths," another cipher, "tenths"; we then place the decimal point: .00054.

As decimals are seldom used beyond the fifth or sixth place, it is convenient to remember the *names* of the decimal orders by the *number of the place*; thus: hundred-thousandths, fifth place; thousandths, third place, etc. If this plan is followed, to write 54 hundred-thousandths, we think at once — the right-hand figure must occupy the fifth place; it will be necessary to prefix three ciphers, so we write at once .00054.

Kinds of decimals. — A decimal may or may not have an integer expressed with the decimal, giving rise to *mixed* and *pure* decimals.

Illustration: 4.05 is a mixed decimal. .54 is a pure decimal.

The figures of a decimal may repeat indefinitely, as .454545. . . . Such a decimal is called a *repeating* or *circulating decimal*, and the group of figures which repeat is called the *repetend*. A decimal may contain a fraction, thus: $.56\frac{1}{3}$. Such a decimal if written in fractional form becomes a complex fraction, $\frac{56\frac{1}{3}}{100}$. It is therefore called a *complex decimal*.

Addition. — The underlying axiom that only like quantities may be added or subtracted is very easy of application in adding and subtracting mixed and pure decimals. Care must be taken to write the decimals in such a way that the tenths will be in one vertical line, the hundredths in another, and so on. We then add the numbers in each column separately as in addition of integers. Thus, to add 4.987, 18.4, and 3.54 we write the numbers as in (a) or (b) (p. 207) and add.

Full explanation. — The sum of the
 (a) (b) thousandths is 7 thousandths; we write
 4.987 4.987 the 7 in thousandths' column; the sum
 18.4 18.400 of the hundredths is 12 hundredths or 1
 3.54 3.540 tenth and 2 hundredths; we write the 2
 26.927 26.927 in the hundredths' place and carry the 1
 to add with the tenths, and so on.

Working explanation. — 7; 12; 6, 10, 19; 4, 12, 16; 2.

In adding complex decimals, a preparatory step is often necessary. In adding $.5\frac{2}{3}$, $.08\frac{5}{8}$, and $9.472\frac{5}{8}$, the fractional parts cannot be added as they stand, because we cannot add $\frac{2}{3}$ of a *tenth*, $\frac{5}{8}$ of a *hundredth*, and $\frac{5}{8}$ of a *thousandth*. We may reduce the $\frac{2}{3}$ of a tenth and $\frac{5}{8}$ of a hundredth to thousandths and then add. The work is below.

$$\begin{array}{rcl}
 .5\frac{2}{3} & = & .566\frac{2}{3} \quad 12 \\
 .08\frac{5}{8} & = & .085\frac{5}{8} \quad 10 \\
 9.472\frac{5}{8} & = & 9.472\frac{5}{8} \quad 15 \\
 \hline
 & & 10.125\frac{1}{8} \quad \frac{37}{8}
 \end{array}$$

Subtraction. — The procedure for subtraction is similar to that for addition. Any one of the three methods, the first or second Italian or the Austrian, may be used.

Multiplication. — Different types of examples arise in multiplication with reference to the term in which the decimal appears. The decimal may be in one term or in both terms. If the decimal occurs in one term only, it may be in the multiplier or in the multiplicand. This gives rise to the multiplication of a decimal by an integer, an integer by a decimal, and a decimal by a decimal.

1. *Multiplication of a decimal by powers of ten.* — To apply the general rule, "To multiply by 10, move each

digit one place to the left" (p. 82), we must move the decimal point one place to the right. To multiply by 100, move the decimal point two places to the right, etc.

2. *Multiplication of a decimal by an integer.* — Take .056 multiplied by 3. This means $.056 + .056 + .056$, because the multiplier 3 shows that the multiplicand is to be used 3 times as an addend. Performing the addition, we obtain .168. Examining the result, we see that the product might have been obtained by multiplying as in integers (168) and then pointing off as many decimal places as there are in the multiplicand (.168). Hence the rule :

To multiply a decimal by an integer multiply as in integers and point off as many decimal places in the result as there are in the multiplicand.

For other methods of development see page 224.

3. *Multiplication by a decimal.* — The simplest type is to multiply by .1, .01, etc. To multiply by $\frac{1}{10}$ is equivalent to dividing by 10, which is done by moving each digit one place to the right (p. 114). This is equivalent to moving the decimal point one place to the left. To divide by 100 move the decimal point two places to the left, etc. Take .056 multiplied by .03. To multiply .056 by .03 is to multiply it by 3 and by $\frac{1}{100}$ since $.03 = 3 \times \frac{1}{100}$. We have just learned how to multiply a decimal by an integer; hence applying that rule, .056 multiplied by 3 = .168; this result must be multiplied by $\frac{1}{100}$; to take $\frac{1}{100}$ of .168 we must move the decimal point two places to the left: .00168. Examining the result, we see that it might have been obtained in an easier way by multiplying as in integers (168) and then pointing off as

many decimal places as there are in both terms, .00168. Hence the general rule, which may apply to any one of the three types :

To multiply decimals multiply as in integers and point off as many decimal places in the product as there are in both multiplier and multiplicand.

Division. — The types of examples in division are similar to those in multiplication ; we may have a decimal in the divisor only, in the dividend only, or in both.

1. *Division of a decimal by an integer.* — Take $3 \overline{) .057}$. To divide 57 thousandths by 3 is to find how many thousandths in 1 of the 3 equal parts of 57 thousandths, just as dividing 57 lb. by 3 means to find how many pounds there are in 1 of the 3 equal parts into which 57 lb. may be divided ; hence, 57 thousandths divided by 3 is 19 thousandths (by the definition of partition). Writing in decimal form, $.057 \div 3 = .019$, which answer may be found by proceeding according to the following rule: To divide a decimal by an integer divide as in integers, and then mark off as many decimal places in the quotient as there are in the dividend.

For a second method of development see page 226.

2. *Division by a decimal.* — Take $.3 \overline{) .0054}$. We know how to divide a decimal by an integer. To make the divisor an integer we multiply it by 10 by moving the decimal point. To keep the value of the quotient unchanged we must make the same change in the dividend, because multiplying both dividend and divisor by the same number does not change the value of the quotient. $\times 3.) \times 0.054$ (Crosses cancel the original decimal points.) We may now apply our rule for dividing a decimal by an

integer: $\times 3.) \times 0.054$. Hence: To divide a decimal (or an integer) by a decimal multiply the dividend and divisor by that power of ten which will make the divisor an integer, and proceed as in the division of a decimal (or an integer) by an integer.

Multiplying and dividing with complex decimals. — The above rules give us the procedure for multiplying and dividing examples in pure or mixed decimals, with a decimal in one or both terms. For multiplying and dividing complex decimals we may modify the above rules to read: Multiply (or divide) *as in mixed numbers* instead of *as in integers*. Otherwise the rules stand unchanged. Thus: (a) Multiply $.013\frac{2}{3}$ by $.8\frac{3}{4}$. (b) Divide $.013\frac{2}{3}$ by $.8\frac{3}{4}$.

(a)	(b)
$ \begin{array}{r} .013\frac{2}{3} \quad 39 \\ .8\frac{3}{4} \quad 16 \\ \hline \frac{1}{2} \quad 6 \\ 9\frac{3}{4} \quad 9 \\ 5\frac{1}{3} \quad 4 \\ 104 \\ \hline .0119\frac{7}{12} \quad \frac{19}{12} \quad \text{Ans.} \end{array} $	$ \begin{array}{r} .8\frac{3}{4}).013\frac{2}{3} \\ 12 \quad 12 \\ \hline 10.5).164 \\ \hline .01\frac{59}{105} \quad \text{Ans.} \\ 10 \times 5.) \times 1.64 \\ \hline 105 \\ \hline 59 \end{array} $

Reductions. — Since a decimal is a kind of fraction, any decimal may be written in fractional form. If the resulting fraction is not in its lowest terms it may be reduced to lowest terms. Hence:

1. To reduce a decimal to a fraction write the decimal in fractional form and reduce to lowest terms, or simplify. Illustration: (a) Reduce .056 to a common fraction.

$.056 = \frac{56}{1000} = \frac{14}{250} = \frac{7}{125}$. (b) Reduce $.5\frac{1}{3}$ to a common fraction. $.5\frac{1}{3} = \frac{16}{10} = \frac{16}{30} = \frac{8}{15}$.

2. The reduction of a fraction to a decimal is less simple. Let us consider the reduction of $\frac{3}{16}$ to a decimal. Since a fraction is an expressed division (see page 157), $\frac{3}{16}$ means divide 3 by 16. 3 units = 30 tenths, which may be written thus, 3.0. We may divide 30 tenths by 16 obtaining 1 tenth with 14 tenths remaining; 14 tenths = 140 hundredths; 140 hundredths $\div$ 16 = 8 hundredths with 12 hundredths remaining, and so on. The work appears below.

$$\begin{array}{r} \frac{3}{16} = ? \text{ dec.} \\ \hline \frac{3}{16} = .1875 \text{ Ans.} \end{array} \qquad \begin{array}{r} .1875 \\ 16 \overline{) 3.0000} \\ \underline{16} \\ 140 \\ \underline{128} \\ 120 \\ \underline{112} \\ 80 \\ \underline{80} \end{array}$$

Circulating Decimals. — If we attempt to reduce $\frac{5}{11}$ to a decimal, we discover that the answer cannot be expressed as a pure decimal. It may be expressed as a complex or a circulating decimal, or the approximate answer may be found to any number of decimal places desired.

$$\begin{array}{r} \frac{5}{11} = ? \text{ dec.} \\ \hline \left. \begin{array}{l} \frac{5}{11} = .45454 \dots \\ \text{or } 454^+ \\ \text{or } .455 \text{ approx.} \end{array} \right\} \text{ Ans.} \end{array} \qquad \begin{array}{r} 11 \overline{) 5.00000} \\ \underline{55} \\ 454 \\ \underline{440} \\ 1000 \\ \underline{990} \\ 1000 \\ \underline{990} \\ 100 \\ \underline{99} \\ 10 \\ \underline{9} \\ 1 \end{array} \qquad \begin{array}{r} 11 \overline{) 5.000} \\ \underline{55} \\ 454 \\ \underline{440} \\ 1000 \\ \underline{990} \\ 1000 \\ \underline{990} \\ 100 \\ \underline{99} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

The circulating decimal is written $\dot{.45}$ and is read: *decimal, repetend 4, repetend 5.*

Two interesting principles. — The question naturally arises, What fractions can be reduced to pure decimals? In reducing $\frac{3}{16}$ to a decimal let us perform the division of 3 by 16 in the form of a cancellation example. The prime factors of 16 are $2 \times 2 \times 2 \times 2$; performing the cancellation we have:

$$\begin{array}{r} .1875 \\ .3750 \\ .7500 \\ 1.5000 \\ \hline 3.0000 \end{array} \quad \cdot$$

$$\frac{3}{16} = \frac{3.0000}{2 \times 2 \times 2 \times 2} = .1875$$

Let us reduce $\frac{7}{125}$ in the same way:

$$\begin{array}{r} .056 \\ .280 \\ 1.400 \\ \hline 7.000 \end{array}$$

$$\frac{7}{125} = \frac{7.000}{5 \times 5 \times 5} = .056$$

Examining the above cases we discover that when the fraction is in its lowest terms, no matter what numerator we may have, if the denominator contains no other prime factor than 2 or 5 we may always annex enough ciphers to cancel all the 2's or 5's or both. This would not be the case if any other prime factor as 3, 7, or 11 appeared in the denominator. Hence, the principle:

Any fraction in its lowest terms which has in its denominator no other prime factor than 2 or 5 may be reduced to a pure decimal. All other fractions will reduce to circulating or complex decimals.

It is helpful not only to know in advance whether a common fraction will reduce to a pure decimal, but how many decimal places there will be in case the result is a pure decimal. Observe the following :

$$\frac{3}{16} = \frac{3}{2^4} = .1875$$

$$\frac{7}{125} = \frac{7}{5^3} = .056$$

$$\frac{7}{40} = \frac{7}{2^3 \times 5} = .175$$

$$\frac{7}{50} = \frac{7}{2 \times 5^2} = .14$$

In the first two examples the exponent in the denominator is the same as the number of decimal places in the result; in the last two examples the higher exponent in the denominator is the same as the number of decimal places in the result. From these instances we may infer the following principle :

*If a common fraction (in lowest terms) is reducible to a pure decimal, the number of decimal places in the result equals the number of times the more frequent factor (2 or 5) of the denominator occurs.*¹

The principle just preceding this one follows as a corollary from the proof; for it is evident that if there were in the denominator of the common fraction any factor other than 2 or 5, it would be impossible to find any integer by which to multiply both terms and get a decimal denominator.

¹ The following is a deductive proof of this principle.

Let $\frac{n}{2^a \times 5^b}$ be any common fraction in lowest terms.

If $a > b$, then, $\frac{n}{2^a \times 5^b} = \frac{n \times 5(a-b)}{2^a \times 5^a} = \frac{n \times 5(a-b)}{10^a}$; because multiplying both terms of a fraction by the same number does not change the value of the fraction.

But $\frac{n \times 5(a-b)}{10^a}$ gives a decimal of a places, because its denominator is 10^a (see definition of decimal, p. 205).

If $a < b$, then, $\frac{n}{2^a \times 5^b} = \frac{n \times 2(b-a)}{2^b \times 5^b} = \frac{n \times 2(b-a)}{10^b}$ which gives a decimal fraction of b places.

In either case the exponent of the more frequent factor in the denominator of the common fraction is equal to the number of decimal places in the decimal equivalent. Hence the rule.

A circulating decimal may be expressed exactly in the form of a complex decimal — in tenths, in hundredths, or in any denomination desired. Thus: $\frac{5}{11} = .4\frac{6}{11}$ or $.45\frac{5}{11}$ or $.454\frac{6}{11}$.

Approximations. — Decimals are useful because of their convenience in solving practical problems. For such purposes, exact answers in the form of circulating or complex decimals are of no particular value. If we were expressing a measure in feet, the answer to the nearest tenth is usually sufficient, as 28.6 ft.; if an amount of money is sought, the answer to the nearest hundredth (cent) is sufficient, and so on.

Let us consider the procedure when approximate answers only are desired. In reducing $\frac{5}{11}$ to a decimal, the answer is $.4^+$. As the next figure to the right is 5, the answer is nearer $.5$ than $.4$. Hence we say $\frac{5}{11} = .5$ approximately. But if the answer were desired in hundredths, we should find that $.454$ is nearer $.45$ than $.46$; hence we write the approximate value of $\frac{5}{11}$ as $.45$ or we may write $\frac{5}{11} = .45^+$.

To add and subtract decimals when approximate answers are sufficient, each decimal should be carried out one place farther than the number of places desired in the answer. The great convenience of using decimals instead of common fractions is seen in such a problem as the following. Find the average per cent of attendance in 3 grades if 97 out of 111 are present in one grade, 89 out of 148 in another grade, and 101 out of 113 in another. Instead of adding the three fractions $\frac{97}{111}$, $\frac{89}{148}$, and $\frac{101}{113}$, dividing by 3 and reducing to hundredths, we may reduce each common fraction to a decimal and then add. Since

the answer is desired true to hundredths, the work is carried out to thousandths.

$\frac{97}{111} = .873$	$.873$	$.893$
$\frac{89}{148} = .601$	$111 \overline{)97.000}$	$113 \overline{)101.000}$
$\frac{111}{111} = .893$	$\underline{888}$	$\underline{904}$
Total $\underline{2.367}$	$\underline{820}$	$\underline{1060}$
Average 79%	$\underline{777}$	$\underline{1017}$
	$\underline{430}$	$\underline{430}$
	$\underline{333}$	$\underline{339}$
	$.601$	$3 \overline{)2.367}$
	$148 \overline{)89.000}$	$\underline{.789}$
	$\underline{888}$	
	$\underline{200}$	
	$\underline{148}$	

For multiplying decimals, when the answer true to any decimal order is required, a convenient procedure is explained below. The explanation given is for practical purposes and the reasons for each step in the procedure are not given. The reader, if interested, may easily figure them out for himself. The illustrations show the successive steps in the procedure. Multiply 428.156 by .062847, finding the answer to hundredths. As in addition and subtraction, it is well to work for one more place in the answer than is required. Hence we shall find the answer to thousandths.

$$\begin{array}{r} 428.156 \\ \times .06.2847 \\ \hline \end{array}$$

$$\begin{array}{r} 4.28,156 \\ \times .06 \ 2847 \\ \hline \end{array}$$

1. Move the decimal point in the multiplier to the right of the first significant figure (the first figure that is not a zero).

2. Move the decimal point the same number of places but in the opposite direction in multiplicand.

$$\begin{array}{r} 4.28\cancel{\times 156} \\ \times 06.2847 \\ \hline \end{array}$$

$$\begin{array}{r} 4.28\cancel{156} \\ \times 06.2847 \\ \hline 25689 \end{array}$$

$$\begin{array}{r} 4.28\cancel{\times 156} \\ \times 06.2847 \\ \hline 25689 \end{array}$$

856

342

17

3

26.907

26.91 *Ans.*

2.8 in the next order. Call this 3.

3. Cross out in the multiplicand all figures beyond the decimal order required (or if necessary, add ciphers to obtain the decimal order required).

4. Multiply by 6, first multiplying the crossed out 5 to determine whether there is anything to carry. Then cross out the 6 and the 1.

5. Multiply by 2, first multiplying the crossed-out figure 1 to see whether there is anything to carry. Place the right-hand figure of this partial product under the right-hand figure of the first partial product. Then cross out the 2 and the 8, and so proceed until all the figures in the multiplicand or multiplier have been used. In the multiplication by 4, 2 (the last figure crossed out) multiplied by 4 = 8; call this 1 to carry; 4 4's are 16; 16 + 1 = 17, the fourth partial product; 7 × 4 (the last figure crossed out) = 28 or

For dividing decimals when an approximate answer is sought, a convenient procedure is explained and illustrated below. Divide 58.921 by 783.469, finding the answer true to hundredths. As in the other operations, we will find the answer to thousandths.

$$7.83\cancel{\times 469} \overline{) 58.921}$$

$$7.83\cancel{\times 469} \overline{) 58.921}$$

$$7.83\cancel{\times 469} \overline{) 58.921}$$

1. Move the decimal point in the divisor to the right of the first significant figure.

2. Move the decimal point in the dividend the same number of places and in the same direction. (Multiplying both dividend and divisor by the same number will not affect quotient.)

3. Retain 3 decimal places in the dividend, crossing out all other figures, because as we are dividing by 7⁺, we shall need as many decimal places in the dividend as we are seeking in the quotient.

$$\begin{array}{r} 7.8\cancel{3}4\cancel{6}9 \overline{).58\cancel{9}2\cancel{1}} \end{array}$$

7

$$\begin{array}{r} 7.8\cancel{3}4\cancel{6}9 \overline{).58\cancel{9}2\cancel{1}} \\ \underline{548} \end{array}$$

.075

$$\begin{array}{r} 7.8\cancel{3}4\cancel{6}9 \overline{).58\cancel{9}2\cancel{1}} \\ \underline{548} \\ 41 \\ \underline{39} \end{array}$$

4. Our new dividend (.589) will require but a two-figure divisor; hence cross out all except the two left-hand figures in the divisor.

5. We now divide 589 by 78, multiplying the 3 that has been crossed out in the divisor by the quotient figure to determine whether there is anything to carry. 7 3's are 21; carry 2. Instead of bringing down the next figure in the dividend, since it is crossed out, we cross out the right-hand figure in the divisor.

6. This procedure is continued until the last left-hand figure of the divisor has been used. Our answer to 3 decimal places has now been found, so we point off 3 decimal places.

Mental computations. — In addition and subtraction, proceed as in mental addition and subtraction of integers. Thus, in $.57 + .96$, the pupil says: .57, 1.47, 1.53 (p. 32). In multiplication and division, if the decimal is equal to a fraction with a small denominator, multiply or divide by the fraction; otherwise proceed as in integers. Thus, if multiplying or dividing by .75, use $\frac{3}{4}$; by .25, use $\frac{1}{4}$; by .625, use $\frac{5}{8}$, and so on. For the complete list of these fractions, see page 391.

HISTORICAL NOTES

The invention of decimals. — It seems almost incredible that the decimal system of writing integers was in use for many centuries before the thinkers of the world hit upon the convenient plan of extending the underlying principle of this system to the right of units. Such a principle had stared more than one mathematician in

the face, as in the case of Cardan's ¹ method of extracting the square root of 10, adopted and used by writers in France and in England. To extract the square root of 10, Cardan first extracted the square root of 10 million, which is 3162; he then expressed $\frac{1}{1000}$ of this answer as shown below.

$$10\ 000\ 000(3|162$$

But no one caught the idea until the 16th century. Simon Stevin of Bruges ² was the first to set forth the theory of the decimal fraction—in an arithmetic published at Leyden in 1585.

Different notations of the decimal.—In the early use of the decimal fraction different notations were used. The decimal point did not come into use for several years after Stevin's work appeared, and even yet has not been universally adopted. The Germans and some of the northern nations of Europe employ the comma for a separatrix.

Stevin used the following forms: ³

$$\begin{array}{cccc} & 0 & 1 & 2 & 3 \\ 5 & 9 & 1 & 2 & \end{array} \text{ or } 5(0)9(1)1(2)2(3) \text{ for } 5.912.$$

Beyer of Frankfort in 1603 published a book in which he used a slightly different form: $\overset{0}{5}.\overset{I}{9}.\overset{II}{1}.\overset{III}{2}$. He also used $\overset{VI}{54}$ for .000054. Oughtred of England in 1631 employed the following form: $5\underline{912}$. As Oughtred's authority was widely recognized, this notation remained in force until the close of the 17th century.⁴

¹ CAJORI, F. — *A History of Elementary Mathematics*, p. 150.

² SMITH, D. E. — *History of Mathematics*, p. 343.

³ CAJORI, *op. cit.*, p. 152.

⁴ CAJORI, *op. cit.*, p. 153.

NOTE.—It is noteworthy that the numerals above, or at the right of, the figures of the decimal are really exponents, indicating in each instance what power of ten is used for denominator; thus we might express 5 units as $\frac{5}{10^0}$, 9 tenths as $\frac{9}{10^1}$, 1 hundredth as $\frac{1}{10^2}$, etc.

Cajori says the decimal fraction in its modern form was first used by Napier; ¹ Ball is inclined to think the honor belongs to Briggs.²

EXERCISES

1. Express entirely in words: 500.007; .507; .0060704; .040 $\frac{2}{3}$; .0 $\frac{2}{3}$; 1.00 $\frac{2}{3}$; .01 $\frac{2}{3}$; £1 = \$4.8665; 1 meter = 39.37079 inches; 1 metric ton = 1.1023 tons; circumference = $3.141592 \times$ diameter.

2. It is often convenient to read decimals by calling the digits; thus, 3.1416 is read *three point one four one six*. Read in this way the numbers in Exercise 1. Should children be taught this method of reading early in their study of decimals? Discuss.

3. Write each of the following decimals in the historic forms shown on page 218: forty-seven hundredths; three hundred six thousandths; twenty-six thousandths.

4. Find the exact sum of (a) $5.62\frac{2}{7}$, $2.234\frac{5}{8}$, and $4.1\frac{2}{3}$; (b) $5\frac{1}{8}$, $3.56\frac{5}{8}$, and $4.0\frac{2}{3}$, expressing the answer in hundredths; (c) three hundred seventeen thousandths, five hundred eighteen thousandths, $7\frac{2}{3}$ tenths, and $.6242\frac{1}{3}$.

5. Find the exact difference between $37.42\frac{2}{7}$ and $18.693\frac{2}{7}$, working (a) with decimals of three orders, (b) with decimals of two orders. Explain the reductions.

6. Find the product of (a) 3800 and 0.043, (b) $6.020\frac{1}{3}$ and 4.7.

7. Divide (a) 14.686 by 700, (b) 0.014686 by .07.

8. Find the quotient to the nearest thousandth of 0.0521 divided by 0.056.

9. The product of two numbers is 0.00045; if one of the numbers is 0.009, what is the other number?

¹ CAJORI, *op. cit.*, p. 153.

² BALL, W. W. — *A Short History of Mathematics*, p. 204.

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10. Reduce (a) $0.0\frac{3}{8}$ to a pure decimal; (b) $0.4\frac{2}{3}$ to a common fraction; (c) \$1 to a decimal of a pound (English); (d) $3\frac{1}{2}$ mills to a decimal of a dollar.

11. Write the list of business fractions (that is, fractions with denominators 2, 3, 4, 5, 6, 8, 10) and their decimal equivalents, expressed in hundredths.

12. Using the above equivalents (a) find the product of 48 multiplied by $.87\frac{1}{2}$, by $.83\frac{1}{3}$, by $.75$; (b) divide 84 by $.33\frac{1}{3}$, by $.83\frac{1}{3}$, by $.87\frac{1}{2}$, by $.75$, by $.66\frac{2}{3}$.

13. What principle of fractions is applied in changing (a) $.6$ to $.600$, (b) $.32$ to $.3\frac{2}{5}$, (c) $0.62\frac{3}{4}$ to $.06242\frac{3}{4}$, (d) $.56\frac{2}{3}$ to $.5\frac{2}{3}$?

14. Find the cost of the following items:

(a) 795 lb. chicken feed @ \$15.75 per ton.

(b) 235 lb. corn meal @ \$1.85 per cwt.

(c) 61500 lb. coal @ \$13.75 per ton.

(d) 9750 shingles @ \$4.35 per M.

(e) 875 feet shelving @ \$83 per M.

(f) 2850 bricks @ \$24.75 per M.

15. A gas meter recorded 17,400 cu. ft. on April 30 and 24,700 cu. ft. on May 31. Find the cost of the gas consumed at \$1.15 per thousand cubic feet.

16. There were delivered at a power plant on a certain day 10 loads of coal weighing in pounds as follows: 5285, 5795, 5640, 5490, 5855, 5720, 6015, 6075, 5905, 6135. Find the cost at \$6.37 $\frac{1}{2}$ per ton.

17. As a result of improved methods of sanitation and food inspection in New York City during the past 25 years the death rate of children under five has fallen from 67 to 25 per 1000; for infants under one year the rate has fallen from 205 to 75 per 1000. Explain how these rates are computed. Express these rates in decimal form and tell what they mean.

18. "The Robert Poole elementary school (in Baltimore), planned for 1088 pupils, cost \$535,587, that being at the rate of 31.2 cents per cubic foot." (Baltimore Bulletin of Education.) Find the number of cubic feet. Find the cost per pupil. Find the number of cubic feet per pupil.

19. The New York City budget for 1923 allows for education and recreation \$89,296,393.98, the total expenses for the year being

\$353,350,975.67. Out of each dollar paid in taxes how many cents is set aside for education and recreation? Use first ordinary division, then the contracted method (p. 216).

20. Using the contracted methods, find (a) the circumference of a circle whose diameter is 76.54 inches, (b) the diameter of a circle whose circumference is 138.65 inches. (Note. Since the measurements given above are correct only to two decimal places, it is useless to carry the computation beyond two decimal places.) The circumference equals 3.1416 times the diameter.

21. Without written work, make a rough approximation of the following: (a) $3.5 \times 5.4 \times 18 \times .33$, (b) $6.3 \times 4.7 \div .3$, (c) $5427 \times 627 \div 943$. How do such estimates make for accuracy in written work? Find the results by written computations and compare with the estimates.

22. Tell which of the following fractions cannot be reduced exactly to pure decimals, and explain why: $\frac{11}{16}$, $\frac{123}{125}$, $\frac{17}{160}$, $\frac{27}{250}$, $\frac{23}{40}$, $\frac{4}{15}$. Tell, in the case of the other fractions, how many decimal places there will be.

23. Reduce to a pure decimal if possible, otherwise to a complex or circulating decimal: (a) $\frac{11}{32}$, (b) $\frac{5}{7}$, (c) $\frac{145}{16}$, (d) $\frac{4}{15}$, (e) $\frac{42}{216}$, (f) $\frac{117}{100}$.

24. (a) Reduce $\frac{4}{3}$ to a circulating decimal, indicating the repetend by dots over the first and last figures. (b) Write the value to the nearest hundredth, to the nearest thousandth. (c) Write the exact value as a four-place decimal.

METHODS OF TEACHING

Alternatives in the choice of method. — We have seen (p. 218) that when decimal fractions were first used, people thought of them as a system of *fractions* convenient for certain purposes, just as the duodecimal fraction of the Romans and the sexagesimal fraction of the Babylonians were convenient, each in its own way. In the early developments of the decimal fraction the point of departure in working out the theory was the common fraction; in the final development it was perceived that

the decimal fraction is a logical extension of the principle of position used in the Hindu notation. In teaching notation of, and operations with, decimal fractions to children we are faced with the question, Which of these two methods of approach shall we use? Shall we make use of the child's knowledge of common fractions, or of his knowledge of reading and writing integers?

In the pages that follow we shall see how these two general methods work out. Some of the processes are more easily understood by the child if related to his knowledge of the decimal notation of integers; others are more easily associated with the corresponding processes with fractions. Without the desire to be dogmatic, we shall suggest certain reasons for choosing one method rather than the other.

Notation. — The children are familiar with the notation of United States money; and this seems to be the apperceptive basis most frequently used in the early lessons in decimals. The following suggests one way of using this basis.

In response to appropriate questions the children pay the \$11.11 amounts shown at the right.

22.22

33.33

etc.

1 ten-dollar bill, 1 dollar bill, 1 dime, 1 cent.

2 ten-dollar bills, 2 dollar bills, 2 dimes, 2 cents, etc.

Then the following relations are expressed.

1 dollar is $\frac{1}{10}$ of 1 eagle, 1 dime is $\frac{1}{10}$ of 1 dollar, 1 cent is $\frac{1}{10}$ of 1 dime.

2 dollars is $\frac{1}{10}$ of 2 tens, 2 dimes is $\frac{1}{10}$ of 2 dollars, 2 cents is $\frac{1}{10}$ of 2 dimes.

Always a figure expresses $\frac{1}{10}$ as much as it would express in the next place to the left.

Since 1 cent is $\frac{1}{100}$ of a dollar, we may read these amounts in this way:

Eleven and eleven hundredths dollars.

Twenty-two and twenty-two hundredths dollars, etc.

Now read the following in the same way, except instead of *dollars* use the name written.

11.11 feet	\$111.1
22.22 feet	35.6 feet
33.33 miles	28.37 feet
44.44 meters	28.07 feet
36.52 feet	etc.
7.75 gallons	

Bring out the following:

We write *tenths* just to the right of the point, we write *hundredths* next to tenths.

In 456.78 the orders are: *hundreds*, *tens*, *units*, *tenths*, *hundredths*; 7 is in *tenths*' order, 5 is in *tens*' order, 8 is in *hundredths*' order, 4 is in *hundreds*' order.

In teaching *thousandths* one may associate it with the mill (thousandth of a dollar), or he may extend the principle of position already developed in the above outline. The higher orders can be taught very directly by applying the principle of position.

The aim of the above development is clear; it is to show how the principle of position (place value) may be used to the right of units. For this purpose we draw on the child's familiarity with writing dollars and cents and, with the help of his knowledge of common fractions, explain the principle (of position) applied in expressing dollars and cents; this principle is then applied with other denominations and with abstract numbers.

Addition and subtraction. — *Difficulties.* — In adding and subtracting integers the “ragged edge” of the example appears, if at all, at the left; with decimals a “ragged edge” may appear at the right also. Practice in writing examples at dictation will help to overcome any such difficulty. Here is seen the importance of knowing the decimal orders by number (p. 206).

Two possible methods. — If a decimal is thought of as a kind of *fraction*, the thought steps in adding or subtracting common fractions would be used in explaining these operations with decimals.

36.200	4.1000
1.453	1.3864
467.230	<u>2.7136</u>
.700	
29.000	.5682
6.009	<u>.2600</u>
540.592	.3082

Reduce the fractions to a common denominator by annexing 0's where necessary, add or subtract the numerators, use in the answer the common denominator. This method was common when the writers were learning the subject.

If a decimal is thought of as an extension of the principle of position to the right of units, we have a more straightforward explanation of these processes, although for convenience, as is common in business, the ciphers may be annexed as above.

Multiplication. — *Multiplying a decimal by an integer.* — The method on page 208 is so simple as to require no further discussion. There are two other methods in common use which we shall discuss briefly.

Regarding a decimal as a fraction, we develop the process as follows:

1. Review multiplying a fraction by a whole number, giving a few examples like: $3 \times \frac{2}{7}$, $4 \times \frac{3}{5}$. Multiplying the numerator multiplies the fraction.

2. Show that the new example is of the same sort ; in the fraction .43, the numerator is 43 and the denominator is 100. It may help to rewrite the example in this form : $\frac{43}{100} \times 7$.

3. Apply the principle in step 1 to the example. Multiplying the numerator by 7, we have as the numerator of the answer 301 ; the denominator of the answer is the same as that in the multiplicand, hence point off two decimal places. It may be necessary to work the example arranged as in step 2. For the purpose of this development do not use a multiplier that contains either 2 or 5 ; for if the form in step 2 is used, the children would cancel, thus distracting the attention from the method desired.

Regarding a decimal as the result of extending the principle of position, we should use the method on page 208, or explain the process as follows : 7 times 3 hundredths are 21 hundredths, which equals 2 tenths and 1 hundredth (write 1 in hundredths' place and carry 2) ; 7 times 4 tenths are 28 tenths, adding the 2 tenths that were carried we have 30 tenths, which equals 3 and 0 tenths.

Multiplying an integer by a decimal. — This type can be explained as a case of finding a fractional part of a number : $\frac{3}{10}$ of 28. Hence the procedure is to multiply by the numerator, 3, and divide the result by the denominator, 10, by moving the decimal point one place to the left.

Or we may think of .3 as $3 \times .1$; to multiply by .3 is to multiply first by 3 and then by .1 (moving the point). See page 208.

Or we may invoke the principle that the product of two numbers is the same whichever is used as the multiplier.

Hence we may regard .3 as our multiplicand, and the answer must have as many decimal places as there are in the multiplicand.

Multiplying a decimal by a decimal. — On page 208 this type is explained in a manner similar to the second explanation given above for multiplying a decimal by an integer.

Another method is to make use of the rule for multiplying a fraction by a fraction. The steps in the first method in multiplying a decimal by an integer suggest the procedure :

1. Review the rule.

2. Show that the example comes under the rule.

3. Apply the rule.

$$\begin{array}{r} .43 \\ .7 \\ \hline .301 \end{array}$$

$$\frac{43}{100} \times \frac{7}{10} = \frac{301}{1000}$$

The details are left to the teacher and student.

Division. — *Types of examples.* — For purposes of teaching it is helpful to distinguish the following types of examples in division of decimals.

I. Dividing a decimal by an integer.

a. Divisor not a multiple of 10. *E.g.*, 3).462

b. Divisor a multiple of 10. *E.g.*, 300)14.7

II. Dividing by a decimal.

a. Dividend having no fewer decimal places than the divisor.

E.g., .06).84, .06).8456

b. Dividend having fewer decimal places than the divisor.

E.g., .043)386, .043)38.6

Methods. — Type Ia is explained on page 209 by regarding the decimal as a concrete number whose denomination is some decimal unit. A variant of this method is to consider the decimal as a fraction and to apply the prin-

ciple, *Dividing the numerator divides the fraction*. In either case the thought is that of dividing the *number* of units and leaving the units unchanged. The manner of presentation is so similar to that for multiplying a decimal by an integer that it is unnecessary to give the details.

Type Ib admits of special treatment. In dividing an integer when the divisor is a multiple of 10 (p. 114) no decimal point was used. Here the decimal point is moved in both terms, dividing by 10, 100, . . . , but no new principle is applied. We

$$\begin{array}{r} 3.00 \times) .38 \times 6 \\ \underline{.128 \frac{2}{3}} \end{array}$$

then have an example of type Ia. Some would defer teaching contracted forms of division until children study decimals. The advantage of this plan is that examples like $300 \overline{)4876}$ and those like $300 \overline{)4.876}$ are treated at the same time and in the same way — by applying the principle, *Dividing both dividend and divisor by the same number does not change the quotient*. It must be borne in mind, however, that if a remainder is desired in the first example, the division must be reversed by a multiplication. In other words, the remainder is not 1.76, but 176.

$$\begin{array}{r} 3.00 \times) 49.76 \times \\ \underline{16} \quad \text{Quotient} \\ 176 \quad \text{Rem.} \end{array}$$

In teaching type II the pupils' knowledge of division by a mixed number is applied. When the divisor is a mixed number, the dividend and divisor are multiplied by the denominator of the divisor; precisely the same thing is done when the divisor is a decimal. In IIa this change gives a dividend which is either a decimal or a whole number, while in IIb the new dividend is always an integer terminating in one or more 0's. The distinction between these two types is wholly unnecessary

when dealing with children of superior, or even average, intelligence ; it is made here merely that the inexperienced teacher may be aware of what may give some children trouble.

Approximations. — How to handle approximations must be learned in connection with the solution of problems, and the more real the problem is to the pupil the more valuable it is in developing judgment as to the significance of figures at the right. The topic is at the best rather difficult and nothing beyond the simplest elements should be attempted during the year in which decimals are first studied. From the first, children can understand that if the answer is the dimension of a garden patch, the value to the nearest tenth of a foot is plenty close enough ; if the answer is money, the nearest cent will do, etc. If the teacher will take the trouble to give the children specific directions as to the number of decimal places to retain in the steps of a complex problem, and will call their attention to the results of neglecting such directions, the children will, by degrees, develop judgment about such matters. In the higher grades a systematic study of the topic *as it applies in practical situations* is well worth the time it costs.

Choice of method. — Considerations of economy of effort in the learning process must at all times determine the teacher's choice of method. Probably the most important principle to observe in this connection is the principle of association of ideas. The teacher seeks, through carefully controlled conditions and activities, to develop certain facts, principles, and skills not as things separate and apart, but as things that cohere to

form large units of knowledge. An organized "hierarchy" of ideas and skills is as important to individual efficiency as an organized social system is to social efficiency.

As to precisely what organization of subject matter is best in a given situation it is impossible to say with assurance, for the question is never a purely logical one. In this chapter we have seen that in teaching decimals the teacher may connect the new work with the child's knowledge of the decimal system of integral notation, or with his knowledge of common fractions. His choice of method must be made according to which of these units of knowledge is the more serviceable, the more thorough and complete, and the more interesting. It is clear that the choice must be made not by the tactician in the rear — the textbook writer or supervisor — but by the subordinate — the teacher — who is directly responsible for leading the advance.

The following suggestions are advanced with the understanding that both units of knowledge mentioned above are sufficiently well known to serve as apperceptive material.

1. In teaching addition and subtraction of decimals, associate with the addition and subtraction of integers. This gives a more straightforward explanation. An example in decimals has the same general appearance as one in whole numbers; and the annexation of 0's may or may not be used.

2. In teaching multiplication and division of decimals, associate with the corresponding processes with common fractions. The work in common fractions is, because of

recency, fresh in mind; the application of the principles of fractions to the operations with decimals affords valuable practice with these principles; the explanations are very direct and simple, each consisting of a single syllogism.

3. In teaching reduction of a decimal to a fraction the fraction concept of a decimal must be used.

4. In teaching reduction of a common fraction to a decimal the extension concept of a decimal is preferred. The thought of reducing to higher terms ($\frac{3}{4} = \frac{?}{100}$) does not lead to the desired process of dividing the numerator by the denominator, and would be very troublesome when the denominator of the given fraction contained some factor other than 2 and 5.

EXERCISES

1. Make a series of examples illustrating the progressive difficulties in addition of decimals. Tell briefly how you would teach each difficulty.

2. In teaching examples like $.91 - .723$ and $.6 - .374$ how would you proceed (a) assuming the children use the Austrian method, (b) assuming they use one of the *take-away* methods?

3. How would you use a list of examples like the following in giving practice in placing the decimal point in the product?

$3 \times .7$, $3 \times .007$, $.3 \times .07$, $.03 \times .07$, $300 \times .007$, etc.

4. Plan a similar lesson for drill on moving the points in division by a decimal.

5. Using $\frac{3}{8}$, explain rationally (deductively) the process of reducing a common fraction to a decimal, as you might to a class.

6. Explain in the same way (deductively) the reduction of $.325$ to a common fraction, as you might to a class.

7. Explain this example in addition, using (a) the fraction concept of a decimal, (b) the *extension* concept, as you would explain in teaching it.

.4
.875
<u>.46</u>

8. Show by a diagram that $\frac{3}{4}$ (of 1) = $\frac{1}{4}$ of 3 or $3 \div 4$. How does this fact help in understanding the reduction of a common fraction to a decimal, from the teacher's point of view? Would it be suitable to present to a pupil?

9. Plan a lesson on the topic: How shall we proceed when we wish to obtain our answer to the nearest tenth, or hundredth, etc.? (a) In addition or subtraction; (b) in multiplication; (c) in division.

10. Of the exercises appearing on pages 219-221 which were intended to fix in mind the fundamental principles of decimals? Which illustrate practical uses of decimals? Which would be suitable for use in grade 6, grade 7, grade 8?

CHAPTER IX
POWERS, ROOTS, EXPONENTS
TEACHER'S KNOWLEDGE
DEVELOPMENT OF SUBJECT MATTER

Origin of the processes. — In the first chapters of this book we have considered the origin and meanings of the so-called *fundamental* operations. We have learned that the first direct process, addition, is a special form of counting and that multiplication, the second direct process, is a special form of addition. We have also learned that each of these processes has its inverse or inverses. The question naturally arises: Is there any other direct process arising as a special form of multiplication and does it likewise have its inverses? The answer is *Yes* and the purpose of this chapter is to explain these new processes.

Involution defined. — A number may be multiplied by itself; this result may be again multiplied by the original number and so on, thus $2 \times 2 \times 2 \times 2 = 16$, in which 2 is used 4 times as a *factor*. This new process is called *involution*, which may be defined as follows:

Involution is the process of finding the result when a number is used a given number of times as a factor. A shorter way of expressing the fact that 2 is used 4 times as a factor is the following: $2^4 = 16$, which is read *2 to the fourth power equals 16*, or, *the fourth power of 2 equals 16*.

The terms in involution are *base*, *exponent*, and *power*. The *base* is the number which is to be used a given number of times as a factor; the *exponent* is the number which shows how many times the base is to be used as a factor; the *power* is the result obtained by using a number a given number of times as a factor. In the above illustration, 2 is the base, 4 the exponent, and 16 the power. Hence, involution is often spoken of as the process of raising a number to a given power.

Evolution defined. — Let us now consider the inverse operations. In the equation, $2^4 = 16$, we have three terms, 2, 4, 16, any one of which may be wanting. If the 2 is wanting, the question is: What number raised to the fourth power equals 16? $x^4 = 16$. That is, we are to find what number used a given number of times as a factor produces a given result. This process is known as *evolution*. To *e-volve* the number thus used is referred to as *extracting the root* of the number, and the customary form of indicating that this operation is to be performed is: $\sqrt[4]{16} = ?$ This is read *the fourth root of sixteen equals what?* The terms are the number or *base*, *index*, and *root*. The *index* is the number which shows how many times a required number is to be used as a factor, or it indicates the root to be extracted. The *base* is the number whose root is to be extracted. The *root* is the number which used a given number of times as a factor will produce a given result, or, it is the result obtained by extracting an indicated root. In the above illustration, 4 is the index; 16 is the number or base; 2 is the required root. In expressing square root it is customary to omit the index; thus, the square root of 25 is written: $\sqrt{25}$.

Logarition¹ defined. — If the 4 is wanting in the above equation the question is: To what power must 2 be raised to produce 16 as the result?

$$2^x = 16.$$

The process arising from this question we may call *logarition*,¹ which may be thus defined: *Logarition* is the process of finding to what power a given number must be raised to produce a given result. The form of expressing this question is: $\log_2 16 = ?$, which is read, "The logarithm of 16 to the base 2 equals what?" The result is called the *logarithm* of the number; 2 is called the *base*; 16, the *antilogarithm* of the number to be found. Hence, *logarition* is the process of finding the logarithm of a number. The customary or common base is 10; in speaking of the logarithm of a number, unless otherwise specified, we mean to what power must 10 be raised to produce the given number. Illustration: $10^2 = 100$; hence the logarithm of 100 is 2; $10^{-1} = \frac{1}{10}$; hence the logarithm of $\frac{1}{10}$ is -1 , and so on. Since 10 is understood to be the base it is not written. $\text{Log } 100 = 2$ is the customary form, not $\log_{10} 100 = 2$.

Methods of finding powers. — We perform the indicated multiplications. Thus, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. The work may sometimes be shortened; thus, in finding the value of 2^{10} we may find the value of 2^5 to be 32 and multiply this answer by itself: $32 \times 32 = 1024$.

We have very little practical need for involution in the

¹ The term *logarition* is here suggested as a suitable name for the process of finding a logarithm. However, the writers do not presume to urge its use; if it satisfies a need, its use will spread. The term was originated by one of the authors years ago, adopted by M. A. Bailey, and used in his *Teaching Arithmetic*.

subject of arithmetic. An understanding of the meaning of the process is necessary to a grasp of the principle involved in compound interest. The algebraic method of raising a binomial to any power should also be understood before the method of extracting roots, explained in succeeding paragraphs, can be properly appreciated. Tables of squares and cubes are convenient to use in finding powers.

Methods of extracting roots. — The practical need for extracting roots arises when we wish to find one edge of a cube of known volume, one side of a square of known area, or the hypotenuse or leg of a right triangle. Such problems, and others involving roots, arise in the measurement of geometric figures. Let us consider some methods of computing roots.

1. *Square and cube roots may be found from tables.* See pages 242, 243.

2. *Finding roots by factoring.* — This method is possible only when the number is a perfect power. Thus, to extract the cube root of 74088, we resolve it into its prime factors :

$$\begin{array}{r}
 2 \overline{)74088} \\
 2 \overline{)37044} \\
 2 \overline{)18522} \\
 3 \overline{)9261} \\
 3 \overline{)3087} \\
 3 \overline{)1029} \\
 7 \overline{)343} \\
 7 \overline{)49} \\
 7
 \end{array}$$

We find that $74088 = 2^3 \times 3^3 \times 7^3$.

Hence the cube root of $2^3 \times 3^3 \times 7^3$, which is $2 \times 3 \times 7$, or 42, is the result sought.

3. *Finding roots by trial.* — By this method we guess at our answer and test each guess until a satisfactory one is found (p. 244).

4. *Finding roots by formula.* — In using this method we regard the number of which the root is to be found as the corresponding power of a binomial. The following examples show the method applied to square and cube root.

Let us extract the square root of 1369.

Let us first consider how many figures there will be in our root. If we square a number of 1 order, we discover that the result is a number of 1 or 2 orders: $1^2 = 1$, $2^2 = 4$, $9^2 = 81$; if we square a number of tens, we shall have a number of 3 or 4 orders: $10^2 = 100$, $99^2 = 9801$, etc. It follows that in the square root of a number there will be as many figures as there are periods of 2 figures each in the number whose square root is to be extracted. Hence, in extracting the square root of a number, we first point off into periods of 2 figures each, thus, 13'69.

Any number may be separated into a number of tens plus a number of units and may therefore be expressed as a binomial of the form $t + u$, where t represents the value of the tens and u the units. Thus in 372, we have 37 tens + 2 units or $370 + 2$. Here $t = 370$ and $u = 2$. Hence, the square, which is our given number, is found from squaring $t + u$. That is, $(t + u)^2 = t^2 + 2tu + u^2$. Our task then becomes: Given $t^2 + 2tu + u^2$, to find $t + u$. We may break up this formula into two parts one of which consists of an expression involving t only, and the other an expression consisting of two factors, one of which is u , thus, $t^2 + (2t + u)u$.

Since t equals the value of the tens, t^2 equals the largest square in the left-hand period. $\sqrt{13} = 3+$; hence $\sqrt{1300} = 30+$, or $t = 30$.

I

	13'69 3	
$t^2 =$	900	
$2t = 2 \times 30 = 60$	469	$t = 30$
$u = 7$		$u = 7$
$2t + u = 67$	469	

The greater part of 469 is $2tu$ since 2 times a number of tens will give a greater result than u^2 (a number of units, squared). Hence if we divide 469 the approximate value of $2tu$ by $2t$, or 60, we shall obtain u (approximately).

II

$$\begin{array}{r}
 13'69 \overline{)37} \\
 \underline{900} \quad t = 30 \\
 67 \overline{)469} \quad u = 7 \\
 \underline{469}
 \end{array}$$

$469 \div 60 = 7$. Hence $u = 7$.
 Adding u to $2t$ we obtain,
 $(2t + u) = 67$. Multiplying this re-
 sult by 7, i.e., by u to get $2tu + u^2$,
 we obtain 469.

These steps are usually combined as shown in II, which is the customary arrangement for the work.

To extract the square root of a number of more than 2 periods we extract the square root of the first two left-hand periods; regarding this result as our new value of t , we then complete the work for extracting the square root of the first *three* periods by finding the new units' figure in the same way as the first units' figure was found. Illustration: Find the square root of 80,656.

$$\begin{array}{r}
 8'06'56 \overline{)284} \\
 \underline{4} \\
 48 \overline{)406} \quad t = 20 \\
 \underline{384} \quad u = 8 \\
 564 \overline{)2256} \quad t = 280 \\
 \underline{2256} \quad u = 4
 \end{array}$$

Explanation.—We first find the square root of 806, which is 28; hence our new t is 280. To find our new u , we divide the new remainder, 2256, by $2t$, or 2×280 ; $2256 \div 560 = 4$ approximately. $560 + 4 = 564$; $564 \times 4 = 2256$; therefore 4 is correct.

Let us extract the cube root of 50,653. The cube of a number of tens is a number of thousands; the cube of a number of hundreds is a number of millions, etc. Thus $10^3 = 1000$; $99^3 = 970,299$; $100^3 = 1,000,000$, etc. Hence, there will be as many figures in the cube root of a number as there are three-place periods in the number. Accordingly, our first step in extracting the cube root of a number is to separate the number into periods of 3 figures each. Thus: 50' 653.

The cube of $t + u$ is equal to $t^3 + 3t^2u + 3tu^2 + u^3$.

Hence our task in extracting cube root is to find $t + u$ from $t^3 + 3 t^2 u + 3 t u^2 + u^3$.

Writing our formula as a guide, and separating it into two parts, as for square root, we have :

$$\begin{aligned}(t + u)^3 &= t^3 + 3 t^2 u + 3 t u^2 + u^3 \\ &= t^3 + (3 t^2 + 3 t u + u^2) u\end{aligned}$$

I

$$\begin{array}{r} 50'653 \quad | \quad 3 \\ t^3 = 27 \ 000 \quad t = 30 \\ 3 t^2 u + 3 t u^2 + u^3 \quad \left. \vphantom{\begin{array}{l} 3 t^2 u + 3 t u^2 + u^3 \\ \text{or } (3 t^2 + 3 t u + u^2) u} \right\} = 23 \ 653 \\ \text{or } (3 t^2 + 3 t u + u^2) u \end{array}$$

II

$$\begin{array}{r} 50'653 \quad | \quad 37 \\ 27 \\ 3 t^2 = 2700 \\ 3 t u = 630 \\ u^2 = 49 \\ \hline 3379 \end{array} \quad \begin{array}{l} 23653 \quad t = 30 \\ \quad u = 7 \\ \hline 23653 \end{array}$$

Since t equals the value of the tens, t^3 equals the largest cube in the left-hand period. $\sqrt[3]{50} = 3^+$; hence $\sqrt[3]{50,000} = 30^+$. $t = 30$.

The greater part of the remainder, 23,653, is $3 t^2 u$ (since 3 times the square of the tens multiplied by u will give a greater result than 3 times the tens multiplied by the u^2 plus u^3). Hence if we divide 23,653, which

is approximately $3 t^2 u$ (as we have just seen), by $3 t^2$, or 2700, we shall obtain u (approximately). $23,653 \div 2700 = 7^+$. Hence $u = 7$. To test this result we add $3 t u$ ($3 \times 30 \times 7 = 630$) and u^2 (49) to 2700, then multiply this result (3379) by 7, which gives 23,653.

The steps are usually combined as shown in II above, which is the customary arrangement.

To extract the cube root of a number of more than two periods we extract the cube root of the first two left-hand periods; regarding this result as our new value of t , we then complete the work for extracting the cube root of the first three periods by finding the new units' figure in the same way as the first units' figure was found; etc. Illustration: What is the cube root of 52,313,624?

	52'313'624	374
	27	
$3 t^2 = 2700$	25313	$t = 30$
$3 tu = 630$		$u = 7$
$u^2 = 49$		
3379	23653	
$3 t^2 = 410700$	1660624	
$3 tu = 4440$		$t = 370$
$u^2 = 16$		$u = 4$
415156	1660624	

Explanation.—We first find the cube root of 52'313, which is 37; hence our new t is 370. To find our new u , we divide the new remainder, 1,660,624, by $3 t^2$, or 410,700; this gives 4 (approximately). $3 t^2 + 3 tu + u^2 = 410,700 + 4400 + 16 = 415,156$; $415,156 \times 4 = 1,660,624$; therefore 4 is correct.

5. *Other roots.*—To find the fourth root of a number we might raise $(t + u)$ to the fourth power, and by applying the formula, proceed as in square and cube root; but an easier way is to extract the square root of the square root of the number. To extract the sixth root of a number we may take the cube root of the number, and the square root of the result; we may proceed in a similar way for any root the index of which has no other prime factors than 2 and 3. Thus $\sqrt[8]{\text{number}} = \sqrt{\sqrt{\sqrt{\text{number}}}}$, and so on.

To find the fifth root, we raise $(t + u)$ to the fifth power obtaining $t^5 + 5 t^4 u + 10 t^3 u^2 + 10 t^2 u^3 + 5 t u^4 + u^5$, and write the expression thus: $t^5 + (5 t^4 + 10 t^3 u + 10 t^2 u^2 + 5 t u^3 + u^4)u$. We find t by extracting the fifth root of t^5 or the left-hand period; we find u , by dividing the remainder by $5 t^4$, and we complete the work by multiplying the numerical values for the expression within the parenthesis by the value of u . The original number must be separated into periods of 5 figures each. Other roots whose indices are prime numbers are found in a similar way.

Roots of fractions and decimals. — To extract the square root of a fraction, (1) we may extract the square root of each term separately, and, if necessary, divide the resulting numerator by the denominator. Thus: $\sqrt{\frac{4}{9}} = \frac{2}{3}$;

$\sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3} = \frac{2.23}{3} = .74+$ or (2) we may reduce the frac-

tion to a decimal and extract the square root of the resulting decimal. To extract the square root of a decimal, we point off the decimal into periods of two figures each, *beginning at the left*. The reason for this may be seen by observing that the square of .1 is .01; of .01 is .0001, and so on. After pointing off the decimal into periods, we proceed as in extracting the square root of integers.

Thus: $\sqrt{\frac{3}{7}} = \sqrt{.42857+} = .65+ \quad .42'85 \quad |.65+$
36
125 6 85
6 25

HISTORICAL NOTES

Involution and evolution. — The process of extracting square root is a very ancient one, dating back as far as 2000 B.C., as is shown from the discovery of the Nippur tablets which contain tables of squares and square roots.¹

The Hindus knew the formula method for finding square and cube roots. For surds (roots of imperfect powers) they found the answer to the n th place by annexing $2n$ ciphers to the number, and writing the result as the numerator of a fraction whose denominator was 1 followed by n ciphers. This method was afterwards used by Cardan (p. 218).

¹ SMITH, D. E. — *History of Mathematics*, vol. I, p. 40.

After the introduction of sexagesimals (numbers with radix 60) from Babylonia, the Greeks used a method of extracting square root similar to our own.

Logarithms. — Logarithms were invented by Napier in 1614. Henry Briggs, an able mathematician of that time, recognized at once the importance of the invention, and made a visit to Napier in 1616. Briggs suggested at this time the use of 10 as a base instead of the base which Napier used. Napier used as base e , whose approximate value is 2.718. Napier approved of 10 as a base, the idea having already occurred to him. Briggs thereafter devoted his energy to the construction of logarithmic tables.

A Swiss genius, Bürgi,¹ is said to have invented logarithms independently of Napier. He was led to the idea through the theory of exponents. In 1620 he published a table of anti-logarithms.

The first natural logarithms were printed in an appendix to Logarithms in 1618 by Oughtred.²

The slide rule. — The slide rule in its modern form was invented by Lieutenant Amédée Mannheim, a French artillery officer, in 1859. This instrument has undergone many mechanical improvements and is made in several sizes and forms with a view to adapting it to various special uses.³ During the past forty years its use has increased tremendously until now it is considered practically indispensable wherever there is a great deal of

¹ BALL, W. W. R. — *A Short History of Mathematics*, p. 202.

² William Oughtred (1574–1660) was an English mathematician. Besides his work in logarithms his invention of the first slide rule entitles him to enduring fame. SMITH, D. E. — *History of Mathematics*, p. 394.

³ BRECKENRIDGE, WM. A. — *Self Teaching Manual*; Keuffel and Esser Company, New York City.

TABLE OF ROOTS AND POWERS

NUMBER	SQUARES	CUBES	SQUARE ROOTS	CUBE ROOTS
1	1	1	1.000	1.000
2	4	8	1.414	1.259
3	9	27	1.732	1.442
4	16	64	2.000	1.587
5	25	125	2.236	1.709
6	36	216	2.449	1.817
7	49	343	2.645	1.912
8	64	512	2.828	2.000
9	81	729	3.000	2.080
10	100	1,000	3.162	2.154
11	121	1,331	3.316	2.223
12	144	1,728	3.464	2.289
13	169	2,197	3.605	2.351
14	196	2,744	3.741	2.410
15	225	3,375	3.872	2.466
16	256	4,096	4.000	2.519
17	289	4,913	4.123	2.571
18	324	5,832	4.242	2.620
19	361	6,859	4.358	2.668
20	400	8,000	4.472	2.714
21	441	9,261	4.582	2.758
22	484	10,648	4.690	2.802
23	529	12,167	4.795	2.843
24	576	13,824	4.898	2.884
25	625	15,625	5.000	2.924
26	676	17,576	5.099	2.962
27	729	19,683	5.196	3.000
28	784	21,952	5.291	3.036
29	841	24,389	5.385	3.072
30	900	27,000	5.477	3.107
31	961	29,791	5.567	3.141
32	1,024	32,768	5.656	3.174
33	1,089	35,937	5.744	3.207
34	1,156	39,304	5.830	3.239
35	1,225	42,875	5.916	3.271
36	1,296	46,656	6.000	3.301
37	1,369	50,653	6.083	3.332
38	1,444	54,872	6.164	3.361
39	1,521	59,319	6.244	3.391
40	1,600	64,000	6.324	3.419
41	1,681	68,921	6.403	3.448
42	1,764	74,088	6.480	3.476
43	1,849	79,507	6.557	3.503
44	1,936	85,184	6.633	3.530
45	2,025	91,125	6.708	3.556
46	2,116	97,336	6.782	3.583
47	2,209	103,823	6.855	3.608
48	2,304	110,592	6.928	3.634
49	2,401	117,649	7.000	3.659
50	2,500	125,000	7.071	3.684

TABLE OF ROOTS AND POWERS — *Continued*

NUMBER	SQUARES	CUBES	SQUARE ROOTS	CUBE ROOTS
51	2,601	132,651	7.141	3.708
52	2,704	140,608	7.211	3.732
53	2,809	148,877	7.280	3.756
54	2,916	157,464	7.348	3.779
55	3,025	166,375	7.416	3.802
56	3,136	175,616	7.483	3.825
57	3,249	185,193	7.549	3.848
58	3,364	195,112	7.615	3.870
59	3,481	205,379	7.681	3.892
60	3,600	216,000	7.745	3.914
61	3,721	226,981	7.810	3.936
62	3,844	238,328	7.874	3.957
63	3,969	250,047	7.937	3.979
64	4,096	262,144	8.000	4.000
65	4,225	274,625	8.062	4.020
66	4,356	287,496	8.124	4.041
67	4,489	300,763	8.185	4.061
68	4,624	314,432	8.246	4.081
69	4,761	328,509	8.306	4.101
70	4,900	343,000	8.366	4.121
71	5,041	357,911	8.426	4.140
72	5,184	373,248	8.485	4.160
73	5,329	389,017	8.544	4.179
74	5,476	405,224	8.602	4.198
75	5,625	421,875	8.660	4.217
76	5,776	438,976	8.717	4.235
77	5,929	456,533	8.774	4.254
78	6,084	474,552	8.831	4.272
79	6,241	493,039	8.888	4.290
80	6,400	512,000	8.944	4.308
81	6,561	531,441	9.000	4.326
82	6,724	551,368	9.055	4.344
83	6,889	571,787	9.110	4.362
84	7,056	592,704	9.165	4.379
85	7,225	614,125	9.219	4.396
86	7,396	636,056	9.273	4.414
87	7,569	658,503	9.327	4.431
88	7,744	681,472	9.380	4.447
89	7,921	704,969	9.433	4.464
90	8,100	729,000	9.486	4.481
91	8,281	753,571	9.539	4.497
92	8,464	778,688	9.591	4.514
93	8,649	804,357	9.643	4.530
94	8,836	830,584	9.695	4.546
95	9,025	857,375	9.746	4.562
96	9,216	884,736	9.797	4.578
97	9,409	912,673	9.848	4.594
98	9,604	941,192	9.899	4.610
99	9,801	970,299	9.949	4.626
100	10,000	1,000,000	10.000	4.641

computing (other than addition and subtraction) to be done. The slide rule is used for performing mechanically any operations which can be performed by logarithms — multiplication, division, involution, evolution, logarition.

EXERCISES

1. In the following, find the value of x : $2^x = 8$, $3^x = 81$, $\sqrt[2]{32} = 2$, $\sqrt[2]{64} = 4$, $\sqrt[2]{64} = 2$, $\sqrt[2]{64} = 8$, $x^3 = 125$, $x^6 = 729$, $x^3 = 216$.
2. Find the values of the following, carrying the answer to two decimal places: (a) $\sqrt{7}$, (b) $\sqrt{.036}$, (c) $\sqrt[3]{54.37}$.
3. Find the values of the following, by first reducing to a decimal and then extracting the root: (a) $\sqrt{\frac{2}{3}}$, (b) $\sqrt{\frac{5}{8}}$, (c) $\sqrt[3]{\frac{3}{4}}$, (d) $\sqrt[3]{\frac{11}{15}}$.
4. Using the tables on pages 242, 243, find the values of the expressions in Exercise 2.

METHODS OF TEACHING

Teaching value of this chapter. — The subjects treated in this chapter are those suitable for use in Junior High School grades. The authors have made no attempt to elaborate this work for schools which have well-defined Junior High School courses. Such schools have access to any of several excellent texts designed exclusively for the Junior High School. This chapter will serve its purpose if it proves helpful to those teachers who wish to introduce in seventh- or eighth-grade classes of superior ability who are not using Junior High School texts some of the subjects recommended by the Committee on National Requirements and used in the more modern courses.

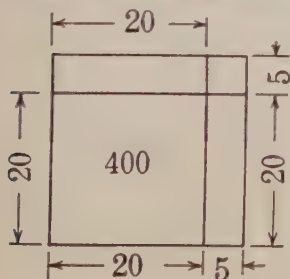
Square root by trial. — In geometrical measurements, the square is a common figure, and exercises in finding its area given one side, and the inverse exercise of finding one side given its area, are valuable. In such exercises

the method of finding the approximate square root by trial is interesting and full of meaning to the child. The work should be introduced by mental exercises. The following illustrates a possible procedure.

A small square of paper 8 in. on each side is to be used for constructing a little basket. How many square inches in it? If there are 64 sq. in. in this square of paper, what will one side measure? If there are 49 sq. in? Suppose there were 30 sq. in. in this square, what would one side be? Yes, more than 5 but not quite 6. Suppose we try 5.5. Let us *square* 5.5; we obtain 30.25. Is 5.5 right? No, it is a little too much. Try 5.4; that is not so good a guess as 5.5, is it? And so on. The work may be carried out to as many decimal places as desired. Thus: Try $(5.48)^2$, and see whether this answer is satisfactory for the square root of 30.

Concrete picture method for square root. — More and more the tendency is growing to enrich an algebraic expression by its geometrical interpretation, and, conversely, to symbolize geometrical relations in algebraic formulas. Hence in giving meaning to the formulas for square root, the concrete method may be interesting. The following illustrations show the geometrical interpretation of the formula for square root.

Let us find the square root of 625.



$$(20 + 5)^2 = 20^2 + 2 \times 20 \times 5 + 5^2$$

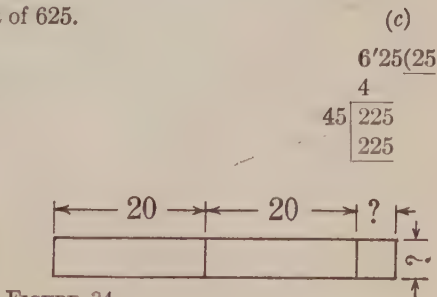


FIGURE 34

(c)

$$\begin{array}{r} 6'25(25 \\ 45 \overline{)225} \\ \underline{225} \end{array}$$

625 is the number of square units in the whole square; the largest square in 625 which is the square of a number of tens is 400, the square of 2 tens or 20. (The square of 30 is too large.) Taking out 400 we have 225 square units left. This is made up of 3 pieces, the approximate length of which (see Fig. 34) is 40; to find the width we divide 225 by 40, giving us 5. Hence the true length is $20 + 5$. Multiplying by 5 we obtain 225, the remaining number of square units. The arrangement of the work is shown in (c).

Square and cube roots by tables. — People who have frequent need for finding square and cube roots, and who do not have to obtain extremely accurate results, use a table like the one on page 242. Any root of two figures only can be read from the table; for other numbers the process of *interpolation* is used.

Find $\sqrt{37.5}$.

Take the square roots of the integers	$\sqrt{37} = 6.083$
just below and just above 37.5, that is, 37	$\sqrt{38} = 6.164$
and 38. Find their difference (.081).	Difference .081
Multiply this difference by the difference	.5
between the given number (37.5) and the	.5 dif. .0405
next lower in the table (37). This is .5,	6.083
and the product so obtained is .0405. In-	Ans. 6.1235
crease 6.083 by this product ($6.083 + .0405$); this result	
is the approximate value of $\sqrt{37.5}$.	

The formula method. — The practical need for extracting square and cube root occurs in problems in mensuration such as finding one edge of a square whose area is known, finding the radius of a circle whose area is given, and finding the hypotenuse of a right triangle.

As the work in mensuration occurs late in the course, the

pupils have presumably had enough elementary algebra to understand the meaning and use of formulas and to perform simple multiplications with literal quantities. They will therefore be able to understand the formula method (p. 236) sufficiently well to use it in any mensuration problems involving extraction of square root. As the formula method is the best method for practical purposes it should receive the chief emphasis.

EXERCISES

1. Children are easily interested in geometric series, *e.g.*, 2, 4, 8, 16, . . . Explain how these might be used in teaching the meaning of the terms *power* and *exponent*.

2. Children get the notion that because $\sqrt{2} = 1.414 \dots$ never terminates the expression has no definite value or meaning. How can you show by a drawing $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.?

3. What would be your dominant aim in teaching the subject of square root — to give an understanding of the nature of the process, or to master the mechanics so as to use the process in a practical way, or to attain both these ends?

4. Make a list of the items developed in this chapter which are worth teaching to every pupil in the eighth year. Are there any additional items that would be suitable as special work for the more gifted pupils?

CHAPTER X

PROBLEMS

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions. — The progress man has made in the understanding of his physical environment has enabled him to exercise increasing control of that environment. The progress he has made in developing and organizing social institutions has promoted general welfare. In both lines of endeavor success has been proportional to man's understanding of *relationships*. Not least among the various kinds of relationships are *quantitative* relationships, the exact expression of which requires the activity of measurement and the use of number. From the construction of a chair or table to the construction of a water system or a sky scraper; from the keeping of a simple expense account to the financial management of a great corporation; from the keeping of a record of school attendance to the taking of the United States census, — in all these activities man is engaged in solving mathematical problems.

All that we have been studying in the preceding chapters about the expression of numbers, the operations with numbers, and the principles which underlie these operations, however interesting they may be to the lover of

mathematics, would have no place in the elementary school course were it not for their practical uses in the affairs of individuals, and of society as a whole. Children must not only be taught to perform the processes but also to use them in practical ways. The more vividly they realize the uses of arithmetic and the advantages of being able to compute with skill and understanding, the more likely they are to be interested and diligent in their practice on the operations.

It follows that in school arithmetic there are two kinds of exercises, those in which the child is told what operation, or operations, to perform, and those in which he must determine for himself what operations to perform. For convenience, the terms *example* and *problem* will be used in this book to distinguish between these two kinds of exercises.

An *example* is a numerical exercise in which the operation is stated; *e.g.*, multiply 37 by 8.

An arithmetical *problem* (*pro*, before; *ballo*, to throw) is a numerical exercise in which the operation is not stated, but may be discovered by reasoning; *e.g.*, Find the cost of 8 chairs at \$37 each.

A problem may involve only one operation or it may involve more than one operation — giving rise to *simple* problems and *complex* problems.

Strictly speaking, the *problem* is the situation itself, and not the *word description* of the situation. The reason why the adult is likely to confuse the two is that he has no difficulty in passing from the description of the problem to the problem itself; but the ability to do this is the result of much first-hand experience with the same, or similar, situations, and of years of practice with text-

book problems. It is of the greatest importance that the teacher bear this in mind in all problem work.

Solution of simple problems. — In every simple problem there are terms whose numerical values are given and a term whose numerical value is to be found ; that is, there is a required term, there are given terms and numerical values for the given terms. The solution of the problem consists in finding the numerical value of the required term. To do this, one must (1) discover the operation which must be performed upon the given term to find the required term, (2) call to mind the numerical value of the given term, and (3) perform the operation upon this numerical value. If we designate the required term by R , the given term by G , and the numerical value by V , the three steps in the solution take this form :

R is thus found from G .

$G = V$.

R is thus found from V .

These three statements constitute a syllogism. The *major premise* (first statement) expresses the relation of R to G , the *minor premise* (second statement) gives the numerical value of G , and the conclusion (third statement) expresses the relation of R to V . To illustrate: What is the cost of 8 chairs at \$37 apiece? The required term is *the cost of 8 chairs*, the given term is *the cost of 1 chair*, and the numerical value of the given term is \$37. The syllogism involved in the solution is, —

Major premise: *The cost of 8 chairs is 8 times the cost of 1 chair.*

Minor premise: *The cost of 1 chair is \$37.*

Conclusion: *The cost of 8 chairs is 8 times \$37, or \$296.*

Analyses, or explanations of simple problems. — While it is true that the above steps must be taken, either consciously or unconsciously, in solving any simple problem, they need not be stated. An explanation, however, of a simple problem must state in substance, if not in form, one or more of these steps.

A *full analysis* gives the entire syllogism, usually in the form of a single sentence, thus: Since the cost of 8 chairs is 8 times the cost of 1 chair, and since the cost of 1 chair is \$37, the cost of 8 chairs is 8 times \$37, or \$296.

If the minor premise is omitted, we have what may be called the *major analysis*,¹ which runs as follows: Since the cost of 8 chairs is 8 times the cost of 1 chair, the cost of 8 chairs is 8 times \$37, or \$296.

If the major premise is omitted, we have what may be called the *minor analysis*, thus: Since the cost of 1 chair is \$37, the cost of 8 chairs is 8 times \$37, or \$296. This minor analysis is the one that has been used so long as the "model analysis" in a large number of textbooks.

If both premises are omitted, we have the *conclusion* only, thus: The cost of 8 chairs is 8 times \$37, or \$296.

The analyses given above are for a multiplication problem. It may be helpful to study illustrations of the *full analysis* of a simple problem in each of the other operations — addition, subtraction, quotition (or measurement), and partition.

Addition. Mary had 10¢ and earned 15¢. What had she then?

Since the final amount is the sum of the original amount and the

¹ This form of analysis appears in *The Pupils' Arithmetic*, by Byrnes, Richman, and Roberts (Macmillan, 1913). See *Book Five* in this series, pp. 10-13.

amount earned, and since the original amount was 10¢ and the amount earned was 15¢, the final amount is the sum of 10¢ and 15¢, or 25¢.

Subtraction. There were 37 children present today; 20 were girls. How many boys were there?

Since the number of boys is the difference between the number of children and the number of girls, and since the number of children is 37 and the number of girls is 20, the number of boys is the difference between 37 and 20, or 17.

Quotition. At \$4 a yard how much material can be bought for \$20?

Since the number of yards bought for \$20 is the number of times the cost of 1 yard is contained in \$20, and since the cost of 1 yard is \$4, the number of yards bought for \$20 is the number of times \$4 is contained in \$20, or 5.

Partition. If a dozen oranges cost 60¢, what is the price of 1 orange?

Since the cost of 1 orange is $\frac{1}{12}$ of the cost of 12 oranges, and since the cost of 12 oranges is 60¢, the cost of 1 orange is $\frac{1}{12}$ of 60¢, or 5¢.

Steps in solving complex problems. — Since a complex problem involves more than one operation, there will be as many component simple problems as there are operations. Hence the solution of a complex problem consists in discovering its component ¹ simple problems and in solving each of these simple problems.

Forms of explanation of complex problems. — (1) A full explanation of a complex problem would consist in stating the first simple problem and giving its full analysis (p. 251), then stating the second simple problem and giving its full analysis, and so on. It is unnecessarily long.

(2) A common form of analysis consists in giving the minor analysis (p. 251) for each of the simple problems, the statement of the problem being omitted. This is a

¹ In advanced work with complex problems the student will often break up a problem into lesser problems which are in themselves complex. The purpose of analysis has been attained just as soon as the problem in question has been separated into problems which are familiar, irrespective of whether these problems are simple or complex.

poor explanation because it presupposes that the simple problems are very easily found, a supposition often contrary to fact.

Illustration: Mary had 50¢; she bought 10 oranges at 3¢ apiece. What had she left? Since 1 orange cost 3¢, 10 oranges will cost 10 times 3¢ or 30¢; since Mary had 50¢ and spent 30¢, she had left the difference between 50¢ and 30¢, or 20¢.

(3) A better form is to state each simple problem and its answer, omitting any analysis of it. This is probably the best explanation because it accounts for the simple problems. Before a child is called upon to solve a complex problem he should certainly be fairly expert in the solution of the simple problems of which the complex problem is composed. This explanation explains the very thing that needs explaining — namely, what the simple problems are; and it omits to explain the easy second step of the solution — namely, the solution of the simple problems.

Illustration: What did Mary pay for 10 oranges at 3¢ apiece? *Ans.*, 30¢. If Mary had 50¢ and spent 30¢, what had she left? *Ans.*, 20¢.

(4) Form (3) may be abbreviated by (a) combining each simple problem and its answer into one statement, or (b) omitting from the statement of the simple problems all mention of the given terms.

Illustrations: (a) The cost of 10 oranges at 3¢ apiece is 30¢. If Mary had 50¢ and spent 30¢, she had left 20¢.

(b) What did Mary spend? *Ans.*, 30¢. What had she left? *Ans.*, 20¢.

(5) The shortest and most advanced explanation consists in stating for each problem the required term and its numerical value.

Illustration : Mary spent 30¢ ; she had left 20¢.

In discovering the component simple problems for a complex problem either of two general procedures may be used, the synthetic or the analytic. In the synthetic our starting point is data given in the problem, from which we proceed to compose simple problems whose results are used as additional data. In the analytic method our starting point is the required term, and our task is to determine what terms are necessary from which to find it. For a detailed treatment of these methods see page 276.

Written solutions. — Some scheme for the arrangement of written work is an aid to orderly thinking and accurate computation. The form here illustrated is such as to be usable in all kinds of problems.

(a) *Simple problem.* — If a man is away from work for 16 days and loses each day his average daily wage of \$8.75, what is his entire loss ?

L. 1 da., \$8.75	8.75
No. da., 16	16
Tot. L., ?	<u>5250</u>
	875
Tot. L., \$140 Ans.	<u>140.00</u>

(b) *Complex problem.* — Mr. Brown sold at one time 150 bushels of wheat at \$1.75 and at another, 80 bushels of corn at \$.68. With part of the proceeds he bought 3 cows at \$95 apiece. How much had he left ?

Sold 1st, 150 bu.	1.75
S. P. 1 bu., \$1.75	<u>150</u>
Sold 2d, 80 bu.	8750
S. P. 1 bu., \$.68	<u>175</u>
Bought, 3 cows	262.50
C. 1 cow, \$95	
Left , ?	.68
	<u>80</u>
S. P. 1st, \$262.50	54.40
S. P. 2d, \$54.40	
Tot. S. P., \$316.90	95
C. Cows, \$285	<u>3</u>
Left, \$31.90 <i>Ans.</i>	285

The short form of explanation is as follows: What is the selling price of the wheat? \$262.50. What is the selling price of the corn? \$54.40. What is the total selling price? \$316.90. What did the cows cost? \$285. What amount was left? \$31.90.

A study of the above arrangement shows that the given terms are written in concise form, followed by their numerical values; that the required term is written with a question mark. This writing of the given and required terms in concise form clarifies the thought and helps to fix the problem in mind. A short line is drawn to separate the statement of the problem from the steps in its solution. Below the line the required term of each of the component simple problems, with its numerical value, when found, is written. This shows the progress of one's thought in solving the problem, as each simple problem must be in mind before the original problem can be solved. It also puts in convenient form a new term which serves in turn as a given term for succeeding simple problems. All scratch work is done at the right, only abstract numbers

being used in performing the operations. This separates the thought part of the problem from the computation and avoids the difficulties that arise when children are required to label all the terms in the operations. The mathematical statements implicit in the examples shown below are of course untrue. Although it is a waste of time to try to explain to children why they are untrue, it would be unwise to permit them to use such forms.

(a)	(b)	(c)	(d)
\$4.25	\$4) \$372	34 ft.	9 sq. ft.) 875 sq. ft.
6 hr.	93 yd.	4 ft.	97 sq. yd. 2 sq. ft.
\$25.50		136 sq. ft.	

The practice of requiring children to indicate every step in the solution in equation form mixes the thought steps with the computation and consumes time needlessly. If a farmer had to solve the problem about the cows, he would merely do the figuring, because the situation is too simple to require reflection. Similarly, when a pupil finds no difficulty in interpreting and analyzing a problem it is easy to dispense with the statements at the left and merely do the scratch work. If the farmer had an unfamiliar or a very complicated problem to work out — such as figuring the cost of material for a silo — he would find it necessary to jot down the items in some such way as is shown on page 255.

Graphic aids. — In many problems, especially those involving the fractional and per cent cases (pp. 177 and 390), the relations may be easily discovered by the use of a graph. Every term mentioned in the problem is so represented that its relation to every other term may

be seen from the graph. The required term may be found from a study of the relations thus disclosed. Illustration: (1) In 2 months a man earned \$270; if what he earned the second month was only $\frac{4}{5}$ of what he earned the first month, what did he earn each month? (2) An article was sold at a gain of $37\frac{1}{2}\%$ of the selling price. If the gain was \$36, what did it cost?

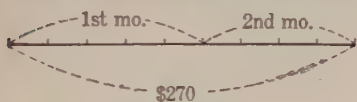


FIGURE 35

9 parts, \$270	
1 part, \$30	
1st sal, \$150	} Ans.
2d sal, \$120	

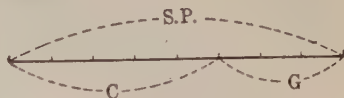


FIGURE 36

3 parts, \$36	
1 part, \$12	
Cost, \$60	Ans.

This does away with such troublesome explanations as "If $\frac{3}{8}$ of the selling price equals \$36, what does $\frac{1}{8}$ of it equal?" to answer which the pupil has to realize that $\frac{1}{8}$ of one thing is $\frac{1}{3}$ of another — always a difficult thing for the child to understand.

The rule and formula methods. — These methods are useful in connection with types of problems that are of rather frequent occurrence — such as those in mensuration, simple interest, and other practical applications. It is wise to avoid rules that cannot be stated briefly and clearly. The formula has the advantage that its use paves the way to the understanding of algebraic equations.

Solution by equation. — Let us first make the distinction between a formula and an equation. A formula is an equation, but strictly speaking an equation is not

necessarily a formula. For example, the formula for finding the area of a rectangle is: $A = b \times h$; but if we know the area and the base and wish to find the height, we must proceed in one of two ways, (1) solve $A = b \times h$ for h and use the result as a formula, or (2) substitute in $A = b \times h$ the numerical values of A and b and then solve for h . Let us illustrate these methods with the problem: Find the height of a sign board 144 sq. ft. in area if its length is 18 feet.

METHOD (1)

$$A = b \times h$$

$$h = \frac{A}{b} \quad \text{Axiom of division}$$

$$h = \frac{144}{18} \quad \text{Substitution}$$

$$h = 8$$

METHOD (2)

$$A = b \times h$$

$$144 = 18 \times h \quad \text{Substitution}$$

$$h = \frac{144}{18} \quad \text{Axiom of division}$$

$$h = 8$$

There is a tendency to introduce the use of equations as early as the 7th year of school. It is a mistake to formalize the work until the children have gained a clear sense of how equations help in solving problems. The following types of equations are all that need be taught in the early work. Later on equations may occur in which two or more types are combined in the same equation.

(1) $x + a = b$, involving the axiom of subtraction, and occurring in problems about *cost*, *gain*, *selling price*, and the like.

(2) $x - a = b$, involving the axiom of addition, and occurring in problems about *cost*, *loss*, *selling price*, and the like.

(3) $ax = b$, involving the axiom of division, and occurring in problems about buying a number of articles at a given price, or about *time*, *rate*, *distance*, etc.

(4) $\frac{1}{a}x = b$, involving the axiom of multiplication, and occurring in problems in finding a whole when a fractional part of it is known.

The language employed in discussing and explaining the solution of equations should emphasize the thought side of the work rather than the mere manipulation of the terms and symbols. For example, such expressions as *transpose*, *canceling like terms from both sides*, *clearing of fractions*, etc., however convenient they may be to those who have mastered the principles, are meaningless to the beginner. Mere mechanical habits are not to be relied upon; but if every equation is solved through the conscious application of understood principles (axioms), the pupil will work always with confidence, and in time he will work rapidly as well.

The axioms can be taught objectively, or by relating them to what the pupil knows about arithmetical relations. Thus, the axiom, *Adding the same number to both sides of an equation gives a new equation (or does not destroy the equality)*, can be taught by weights on a simple balance, or by association with the fact that the minuend is the sum of the subtrahend and remainder (see type (2) above). Similarly, the axiom of division can be taught objectively or by association with the principle, *Dividing the product of two numbers by one of the numbers gives the other number*.

Solution by proportion. — A problem can be solved by proportion if it involves no operation other than multiplication and division.

Problem 1. An automobile travels 246 miles in 8 hours. At this rate how far will it travel in 15 hours? Here, as in every problem in simple proportion, there are two terms — so related that a change in one causes a change in the other. Each term has two values. Both values of one term and one of the values of the other term are given; the problem

is to find the missing value. In this problem the terms are *time* (number of hours) and *distance* (number of miles), the values of time are 8 hours and 15 hours, the given value of distance is 246 miles, and we are to find the missing value of distance. Since *multiplying* the time *multiplies* the distance, the ratio between the first time (8 hours) and the second time (15 hours) will equal the ratio between the first distance (246 miles) and the second distance (x miles). Equating these ratios we have :

$$\text{Form A. —} \quad (1) \quad 8 : 15 = 246 : x.$$

Taking the product of the means (middle terms) equal to the product of the extremes (end terms) we have :

$$(2) \quad 8x = 15 \times 246.$$

Dividing both members of the equation by 8 we obtain :

$$(3) \quad x = \frac{15 \times 246}{8} = 461\frac{1}{4}.$$

$$\text{Form B. —} \quad (1) \quad \frac{8}{15} = \frac{246}{x}.$$

Multiplying both members of the equation by $15x$ we obtain :

$$(2) \quad 8x = 15 \times 246.$$

The rest of the work is in *Form A*.

Problem 2. Traveling at the rate of 25 miles an hour a trip consumes 7 hours. If it is desired to make the return trip in 5 hours, what must be the rate of travel? In this problem the terms are *rate* and *time*, two values for *time* are given but only one value for *rate*. Since *multiplying*

the time *divides* the rate, the ratio between the first time (7 hours) and second time (5 hours) will equal the ratio between the second rate and the first rate. Equating these ratios, we have :

FORM A

$$(1) \ 7 : 5 = x : 25$$

$$(2) \ 5x = 7 \times 25$$

$$(3) \ x = 35$$

FORM B

$$(1) \ \frac{7}{5} = \frac{x}{25}$$

$$(2) \ 5x = 7 \times 25$$

$$(3) \ x = 35$$

It is obvious from the above solutions that what we call proportion is merely a *way of thinking*, employing an appropriate terminology which leads to the expression in equation form of the relation involved in a problem. The expression of the relation is simple enough, once it is found. The important step is to discover the relation, and this is done by asking one's self the question, "What would be the effect upon the second term of multiplying the first term?" If a change in the first term produces the same change in the second term, the relation is said to be *direct*, leading to a *direct proportion*; but if a change in the first term produces the opposite change in the second term, the relation is said to be *inverse*, leading to an *inverse proportion*. Thus, in problem 1 we discovered that *the distance is directly proportional to the time*; and in problem 2 we discovered that *the rate is inversely proportional to the time*.

It appears, then, that the essential thing in a solution by proportion is not the *form* in which the relation is stated in symbols, but, rather, the *mode of thought* back of the symbolic expression. *Form B* is rapidly displacing

Form A, but often the practiced worker will set down neither of these forms. He will think in terms of proportion and merely make the necessary computations.

Checking problems. — The checking of the correctness of a solution involves more than checking the operations involved; the analysis may be wrong. The following methods are useful; the selection of the method is a matter of judgment:

(1) Think the problem through a second time. This is probably the most common method.

(2) If the problem involves an indirect relation, formulate and solve the problem involving the corresponding direct relation. The answer to the *check* problem should be a given term in the original problem.

(3) Study the situation to see if the answer obtained fits the conditions of the problem.

(4) Work the problem using “rounded” numbers and see whether the answers are about the same in value.

(5) Ask yourself if the answer is a reasonable one.

As soon as pupils are old enough to understand problem checking they should be required to follow their explanations of a problem with a brief statement of how they checked their answer. This will give the teacher opportunity to guide his pupils in wise selection of checking methods.

Types of complex problems of two operations. — Theoretically there are as many kinds of two-step problems as there are ways of pairing the five types of simple problems; thus:

Addition — Subtraction
Addition — Multiplication

Multiplication — Addition
Multiplication — Subtraction

Addition — Partition
Addition — Quotition

Multiplication — Partition
Multiplication — Quotition

Subtraction — Addition
Subtraction — Multiplication
Subtraction — Partition
Subtraction — Quotition

Partition — Addition
Partition — Subtraction
Partition — Multiplication
Partition — Quotition

Quotition — Addition
Quotition — Subtraction
Quotition — Multiplication
Quotition — Partition

While it is perfectly true that not all these types occur in practice, most of them do occur. Of those that occur in practical life some are of very frequent occurrence and some are rare. A study of some well-known problem lists has revealed the fact that there is a surplus of problems of certain types and a decided lack of certain other types. This probably does not come about by design — it just happens that way. Some types that are very often used in everyday affairs are almost entirely absent from one list containing several hundred problems. It devolves upon the teacher to give this matter conscious attention. He should determine what types are worth teaching and then see that the problems assigned are selected with a view to insure a fair mastery of useful types without over practicing on any one type. Working a thousand problems of the partition-multiplication type (of very frequent occurrence in most problem lists) will help scarcely at all in coping with the addition-subtraction type.

It would be useless to extend our classification to include problems of three or more steps.

Kinds of problem material. — Some problems involve situations that arise in the affairs of everyday life and some do not, giving rise to the practical or socialized problem and to the recreational or puzzle problem. Of the practical problems some are of a general nature and should be understood by everybody; others involve technicalities which perhaps do not warrant the expenditure of the time necessary to master them, except by persons who need to use them or who are interested in investigation. For example, every one needs some knowledge of solving problems involving operations in integers, fractions, decimals, and common measures, but not every one needs to know how to compute the cost of papering a room or of evaluating sinking funds and annuities.

The recreational problem has the same value in arithmetic that recreation has in any other field of life and should be used, not as something that must be mastered, but as a "quickener of the wit." Many puzzle problems are also of historic interest, and it is as much a matter of general information to know of Achilles and the hare, or of magic squares, as to know of Socrates and the hemlock, or the conquests of Alexander. Such problems as, "If A can do a piece of work in 6 days and B in 8 days, in what time could they do it together?"; "At what time between 5 and 6 o'clock are the hands of a watch together?"; and "If a hound can leap 5 ft. while a hare leaps 3 ft., in how many leaps can the hound overtake the hare, if the latter has a start of 16 ft.?" are all good fun as puzzle problems and should be so used if introduced at all. The first and last named have historic prototypes which may be given in this connection if desired.

HISTORICAL NOTES

Mathematical curiosities. — Problem solving in centuries past seems to have been largely a matter of recreation and diversion. Many of the problems proposed have become famous as historical curiosities. Some of the problems which were of practical interest in the long ago have also become interesting as historical material. Attempts to solve these problems often led to interesting mathematical discoveries.

One of the earliest and likewise one of the best known historical puzzlers is the problem of Achilles and the hare, proposed by Zeno,¹ a Greek philosopher who lived in the fourth century B.C. Tradition ascribes to Euclid the mule and donkey problem.² The Hindu problems are unique because of the flowery language in which they were couched, and because they were proposed as a favorite social amusement. An illustration³ of the kind in which they indulged is the following: "Beautiful maiden with beaming eyes tell me, as thou understandest the right method of inversion, which is the number which multiplied by 3, then increased by $\frac{3}{4}$ of the product, divided by 7, diminished by $\frac{1}{3}$ of the quotient, multiplied by itself, diminished by 52, by extraction of the square root, addition of 8 and division by 10 gives the number 2."

¹ "Zeno argued that if Achilles ran ten times as fast as a tortoise, yet if the tortoise had (say) 1000 yards start it could never be overtaken; for when Achilles had gone the 1000 yards, the tortoise would still be 100 yards in front of him; by the time he had covered these 100 yards, it would still be 10 yards in front of him; and so on forever. Thus Achilles would get nearer and nearer to the tortoise but never overtake him." Ball, W. W., *A Short History of Mathematics*, p. 33.

² "A mule and a donkey were going to market laden with wheat. The mule said, 'If you gave me one measure I could carry twice as much as you, but if I gave you one we should bear equal burdens.'" Ball, *op. cit.*, p. 64.

³ CAJORI, F. — *History of Elementary Mathematics*, p. 92.

An interesting collection of "problems for quickening the mind" dates back to about 1000 A.D. and contains among others the hare and hound problem.¹

Among the Italian mathematicians both Pacioli and Tartaglia gave collections of puzzle problems. Cajori cites the problem of the three jealous husbands who wanted to cross a river in a boat holding only two persons so that none of the three wives would be found in company with one or two men unless her husband was present. He also gives the problem of the 15 Christians and 15 Turks which occurs in Kersey's Wingate: 30 persons were to be placed in a ring in such a way that beginning at one of the persons and proceeding circularly every ninth person should be cast into the sea; arrange them so that all the Christians may be saved.²

Such problems as had to do with the length of time required to fill a cistern which could be filled in so many hours by one pipe and a different number of hours by another pipe, when both pipes were running, was a practical problem among the Greeks who were interested in the construction of public baths, swimming pools, etc. It is the prototype of such problems as: A does a piece of work in 3 days and B in 5 days, in what time will they finish it together?

Cajori³ quotes Bramagupta, who lived about 628 A.D., as saying: "These problems are proposed simply for pleasure; the wise man can invent a thousand others. As the sun eclipses the stars by his brilliancy so the man of knowledge will eclipse the fame of others in assemblies of

¹ CAJORI, F. — *History of Elementary Mathematics*, p. 113.

² *Ibid.*, pp. 220, 221.

³ *Ibid.*, p. 100.

the people if he proposes algebraic problems and still more if he solves them."

Problem contests. — One of the practices in the Middle Ages was to propose to rival mathematicians problems beyond their reach. Public contests were held and great honor attached to the winner. A mathematician was honored and admired for his ability. One of the most famous of these contests was that held on Feb. 22, 1535, in which Tartaglia challenged Floridus to solve several problems involving cubic equations. Tartaglia, who had succeeded in discovering a general method for the solution of cubic equations, solved the problems submitted to him in two hours and won the contest from his rival, who could not solve any of the problems which Tartaglia had proposed to him.¹

EXERCISES

1. Formulate three problems involving *cost*, *gain*, and *selling price* — first, to find the selling price; second, to find the gain; third, to find the cost. Give for each problem (a) the *full analysis*, (b) the *major analysis*, (c) the *minor analysis*, (d) the *conclusion* only.

2. Formulate three problems involving *cost*, *loss*, and *selling price*, and give for each problem the four forms of analysis.

3. Formulate three problems involving *time*, *rate*, and *distance*, and give the four forms of analysis.

4. A haberdasher bought ties at \$9.25 a dozen and retailed them at \$1.75 apiece. Find the gross gain on 25 dozen ties. (a) Give a formal written solution as on page 255. (b) Explain your solution by each of the methods described on pages 252-254.

5. A man timing himself finds that he walks two blocks in three minutes. If there are 20 blocks to a mile, what is his rate in miles per hour? Solve mentally, stating the simple problems and their answers.

¹ CAJORI, F. — *Op. cit.*, p. 225.

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6. A man's income is \$235 a month. If he wishes to save \$400 a year, his average monthly expenses must not exceed what sum? Give the written solution and explain by stating the simple problems.

7. It is estimated (1924) that there are 745,000,000,000 cubic feet of standing timber in the United States, and against this supply there is an annual drain of 25,000,000,000 cubic feet, while the annual timber growth is only 6,000,000,000 cubic feet. At this rate how long will our supply of timber last?

8. Mr. Greeley, chief of the U. S. Forest Service, declares that 6,000,000,000 cubic feet of timber is wasted every year through inefficient methods. Were this needless waste eliminated, how many more years would our national supply last?

9. According to a folder issued in 1924 the Salvation Army collects and returns to the (paper) mills approximately 55,000 tons of paper. In this way "American housewives each year, through The Salvation Army, are conserving approximately 9000 acres of standing timber." What fact can you find from these data? Find it.

10. During the first six months of 1924 there were 654 forest fires in the forests of Washington and Oregon, 132 caused by smokers and 101 by carelessly handled camp fires. These fires swept over 33,500 acres. Bring out the significant facts that can be derived from these data.

11. A New York teacher who lives in a suburb travels on a monthly commutation ticket which can be purchased on any day of the month and is good up to, but not including, the corresponding day of the next month. The commutation ticket costs \$6.05 and a single ticket (one way) costs 21 cents.

(a) Find the saving on a commutation ticket purchased Monday morning, September 8, assuming it is not used on Saturdays and Sundays.

(b) If such a ticket should expire on a Thursday, would it be better to buy single tickets on Friday and a monthly ticket the following Monday morning?

(c) If the monthly ticket should expire on Tuesday, Dec. 9, and the holiday vacation is from Dec. 25 to Jan. 1 inclusive, will it be better to buy a monthly ticket dated Dec. 10, or to use single tickets up to the holidays and get the monthly ticket on Jan. 2? It is assumed that no use of a ticket would be had during the holidays.

(d) What is the latest date in December on which it would be a saving to buy a monthly ticket?

(e) Using the current school year, plan for this teacher on what dates he should buy monthly tickets and on what dates he should use single tickets in order to travel at least expense. Assume that the Christmas vacation is from Dec. 25 to Jan. 1 inclusive, the Easter vacation is the week after Easter, and that on all legal holidays school is closed.

12. In solving the problems in 11 which method of finding the simple problems is better? Why?

13. State the component problems for 11 (a), (b), (c), (d).

14. Could one use the analytic method of thinking in Exercises 9 and 10? Give reason for your answer.

15. Show how you would use graphic aids in solving the following problems:

(a) Stations A and B are 230 miles apart. Two trains start from A and B at the same time and move toward each other; the train from A travels 30 miles an hour and the train from B 25 miles an hour. At what point will they meet, and when?

(b) A trolley line maintains a 30-minute schedule between cities R and C, a car leaving R for C and a car leaving C for R on the hour and half hour. If the trip each way requires $3\frac{1}{2}$ hours, how many cars will I meet on a trip from R to C?

(c) A man who charges 50¢ an hour for sawing wood requires 48 minutes to saw a log into 5 equal pieces. At this rate what would have been the labor cost if he had sawed the log into 4 equal pieces?

(d) In framing a house the *studding* ($2'' \times 4''$ uprights to which the siding is nailed) is placed at intervals of 16 inches *between centers* and each corner is made stronger by using two *studs* nailed together. How many studs will be required for a house 32' by 40'?

16. What is the meaning of *one inch of rain*? How many barrels fall on an acre when there is one inch of rain? How much would this water weigh?

17. In many parts of the country the rain falling on the roofs of houses is conveyed to cisterns. If the horizontal area covered by a roof is 32' by 38' and a shower raised the water level 26'' in a cylindrical cistern 8' in diameter, what was the depth of the rainfall?

SOME PUZZLES

1. A man was setting up a row of posts around a plot of ground. He found that if he set his posts a foot apart, he had 150 too few. He tried setting them a yard apart, but found that he then had seventy left over. How many posts did he have?

2. Given an 8-gallon can full of water and a 5-gallon can and a 3-gallon can, both empty; to divide the water into two equal parts without using any receptacles except the three ungraduated cans.

3. A swindler bought a pair of shoes for \$5, giving in payment a counterfeit \$50 bill. The dealer, not having the change, went next door and obtained it. Later the character of the bill was discovered and the dealer made good the neighbor's loss. How much did the dealer lose?

4. The Joneses drove over to visit the Browns. They found on reaching there that the Browns had been gone 11 minutes to visit the Joneses, so started back home. The Browns found the Joneses had been gone fifteen minutes, so set out to return. The two parties met midway between their homes at 4 o'clock. At what time did the Browns leave their home? It is to be assumed that the two cars maintained their respective rates of speed throughout, and traveled the same route.

METHODS OF TEACHING

TEACHING SIMPLE PROBLEMS

Early beginnings. — The foundation stones for the ability to solve problems are the simple acts of judgment which the child makes in connection with his objective work in the early grades. At first there are the judgments that this is larger or smaller, longer or shorter, etc., than that. These are soon followed by statements of how much larger, or how many times larger, or one-half as large, etc. Every new number combination is worked out objectively, or is imaged in terms of previous objective experience,

or is closely related to some fact that grew out of concrete experience. Such experiences, by stimulating motor and verbal responses in wise combination, lead by easy stages to the more exacting tasks called problem solving. It is distinctly unwise to attempt forcing growth in these things; some children need a great deal of experience in dealing with simple problems which present themselves as situations (not described in words) before they can even tell what they did or are going to do. The first problems to be presented in word form should be such as can be easily related to previous experience, and the language should be the simplest. Categorical statements are simpler than *if* clauses; statements in which the child is expected to fill in a blank are better than questions. For example:

If a boy had only 8 cents in his pocket, how many more cents does he need to buy a ten-cent ball? This can be simplified to:

Henry wanted to buy a ball. The ball was marked 10¢. He found he had 8¢ in his pocket.

He asked his mother for — more cents.

'Pupils' difficulties. — In solving a simple problem a child has to perform two acts of judgment: (1) to determine *what operation* he should use, (2) to decide *what terms* to use in the operation. These are the acts of judgment which find expression in the premises of our syllogism (p. 250). In the case of the conventional textbook problem the selection of the terms usually presents no difficulty, because these terms and their numerical values are nearly always given in the problem. Sometimes the value of a term is missing, as in the problem:

How many yards in 65 feet? in which the fact that 1 yard equals 3 feet must be recalled. Also, in some of our modern arithmetic texts questions like these occur: At present prices find the cost of the following: (a) $2\frac{3}{4}$ pounds of sirloin steak; (b) 3 pounds of sugar; etc. Occasionally a textbook problem will mention terms not necessary to the solution, and this presents the necessity for conscious choice. Such problems, except when they occur naturally, as in projects, are viewed by children as "catch" problems. When used, the teacher should frankly say that he is going to see if he can catch the pupils on some queer problems; then the children will view the work as a game rather than as something unfair.

It is clear that of the two acts of judgment mentioned, the selection of the operation, *i.e.*, the discovery of the relation between the required term and the given term, is the one that will give the most trouble.

Pupils' explanations. — The best teaching practice, relative to explanation of problems, is to require the pupil to give, in a sentence, a straightforward statement of his answer — that is, the *conclusion*. If the conclusion is correct, it is fair to assume that the reasoning has been correct. If the conclusion is false, the teacher will see at once which step in the reasoning is at fault and proceed to set the pupil right. In other words, the formal logic of the syllogism is not something to be taught, but rather a guide to the teacher in sound teaching.

How to help the pupil. — If the pupil has used the wrong operation, it is because he has failed to understand the situation which the problem essays to describe; either the relationship of the terms is novel to the child or else

the wording of the problem is not understood.¹ To clarify any language difficulty, the pupil should be required to restate the problem in his own words and to write down very briefly, and separately, each fact stated in the problem and the question which it asks. The meaning of unfamiliar words should be explained, if necessary. If, on the other hand, the wrong choice of operation is due to inability to see the relation between the terms, it will be necessary to study the problem situation with a view to making it real. Frequently it will be a help to represent the relationship of terms by objects or drawings, and whenever possible the objects should be manipulated, or the drawings made, by the children themselves; what the teacher does may not secure the pupil's attention, but what the pupil does himself he perforce attends to. Sometimes the dramatization of a problem situation will clarify matters. Often a relationship will seem obscure just because the numbers are large — in which case the children should read the problem through, substituting small numbers. After they have solved the latter problem they should be able to solve the original problem.

A child who could not solve the problem "If a kitten in a basket weighs 18 oz. and the basket alone weighs 2 oz., what does the kitten weigh?" was asked to represent the basket by a box and the kitten by an eraser placed in the open box. A high school graduate who explained the problem "What was the cost of an article that sold for \$84 at a loss of \$16?" by saying "The cost was \$84 minus \$16" was asked to work a similar problem using \$6 for the selling price and \$2 for loss.

¹ S. A. Courtis measured the effect of the language element upon the difficulty of a problem. He worded the same problem in 21 different ways. The hardest of these problems, or wordings, as measured by the errors made by children, was found to be nineteen times as difficult as the easiest. *Courtis Standard Tests* (Bulletin No. 2, 1913).

She explained it as she did the first. The following procedure was then tried: Draw a line to represent your cost. Will the selling price be more or less than the cost? Less. Very well; cut off a piece of your line and label it *loss*. What will the part left represent? Selling price. Right. Now mark the numerical values for what you know. The diagram was as follows:

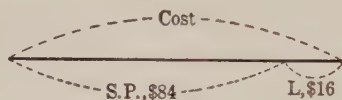


FIGURE 37

Now study your picture and tell me what is the value of the cost. After a prolonged study of the situation, she gave the correct result.

In studying the relation between the required term and the given term it helps the thinking to express each term in definite form. For example, in thinking of the problem on page 251 it is better to talk about "the cost of 8 chairs" than about "the total cost."

Habit in problem solving. — It is generally agreed that no set form of so-called analysis can help the child in solving simple problems. Success in problem solving does not depend upon the formation of any *mechanical* habit nor upon superficial judgments based upon the *sound* of the problem. The only serviceable habit is a correct habit of *thinking* — an orderly procedure in attacking problems. The following procedure has been found helpful:

(1) Center attention on the required term; (2) center attention on what is given; (3) determine how to find the required term from the given term by considering what operation is needed; (4) find the required term by performing this operation. To illustrate, take the quo-

tition problem on page 252. What must I find? Number of yards for \$20. What do I know? Cost of 1 yard. How can I find the number of yards? By finding how many times the cost of 1 yard is contained in \$20. What then is the number of yards? 5.

In certain oft-recurring types it is doubtless a real advantage for the pupils to learn the general statements of the relations. Thus, after they have thought through several concrete situations by enacting the transactions involved, it is good practice to crystallize the experience in such statements as: The cost of a number of articles is that number of times the cost of one article. The cost of one article is $\frac{1}{6}$ the cost of 6 articles. The selling price is the cost plus the profit. The selling price is the cost minus the loss. These general facts enable the pupil to solve subsequent problems without thinking through the situation; they merely recognize the situation as a familiar one, recall the general law (major premise), and apply it.

TEACHING COMPLEX PROBLEMS

Pupils' difficulties. — In solving simple problems the child has to settle the question of what operation to use; this settled, he finds the answer *by one step*. The pupil's initial difficulty with complex problems arises from the fact that he quite naturally expects to get the answer by one step; his experience with problems has furnished a seemingly firm basis for this expectation. The teacher's job is to make the child aware of the nature of the new difficulty, and then to help him to meet it. The explanations given on page 253 are helpful in making the method

of solution clear to others *after the problem has been solved*, but they are of little value in teaching children *how to attack* a complex problem that presents difficulty. What is needed, is systematic instruction and practice in the technique of resolving a complex problem into its component problems. We shall discuss two methods of attacking this difficulty.

Method of synthesis. — One method of procedure in discovering the component simple problems is to select a portion of the problem — a fact or two — and from this form a simple problem; combine the result of this first simple problem with another portion of the original problem and form a second simple problem, and so on. This may be called the *synthetic method*. It sometimes proves to be a more direct, simple, and natural method than any other. Thus:

Mary bought 10 oranges at 3¢ apiece and 3 bananas for 5¢; she gave the fruit dealer 50¢. What change did she receive? Read as far as the word “apiece” and form a simple problem. Mary bought 10 oranges at 3¢ apiece. What did they cost? *Ans.*, 30¢. Now use this result and the next part of the problem as far as the semicolon, and make another simple problem. Mary paid 30¢ for oranges and bought 3 bananas for 5¢. What did she pay for both? *Ans.*, 35¢. Using your last result and the rest of the problem, make another simple problem. Mary paid 35¢ for fruit and gave the dealer 50¢. What change did she receive? *Ans.*, 15¢.

Method of analysis. — Another procedure is to center the attention first on the required term and consider from what this required term can be found; it can be found from one or more other terms. Then question from what each of these terms can be found, and so continue

until terms are reached whose numerical values are known. This may be called the *analytic method*. It cultivates good habits of thought and leads accurately step by step to the desired goal. It sometimes proves a more difficult method than the synthetic as it is not always evident what terms must be known from which the required term can be found.

Suppose, for example, a pupil has difficulty with the following: If Mary paid 24 cents for 3 grapefruit, what would 5 of them cost at the same price? The teacher would guide the pupil's thought somewhat as follows:

What does the problem ask you to find?

The cost of 5 grapefruit.

What do you have to know?

The cost of one grapefruit.

That is right. Now read the problem carefully to see if you can find some way of getting the cost of one grapefruit.

The problem on page 276 would be taught in similar fashion, the questions being framed so as to bring out the following thought steps:

I must find what change I will get out of 50 cents.

I must know the cost of all the fruit.

I can find the cost of all the fruit from the cost of the oranges and the cost of the bananas. I know the cost of the bananas.

I can find the cost of the oranges from the number of oranges and the cost of one.

It is sometimes helpful to crystallize the above procedure in diagrammatic form. The number of simple problems involved in any complex problem and the dependence

of each term upon others is clearly shown in the *analytic diagram* below.

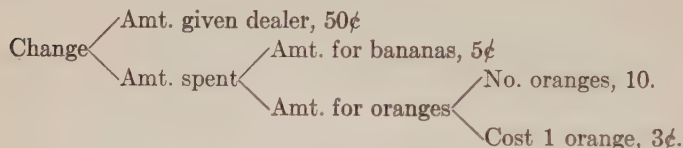


FIGURE 38

It is read thus: The required term is the *change*; this can be found from the amount given the dealer and the amount spent; the amount spent can be found from, etc.

Methods of conducting class work in problems. — The following suggestions for the conduct of class work in problems should prove helpful:

1. Before assigning problems in the textbook read critically, and eliminate those that for any reason are unsuited to the children.

2. A lesson on problems, like any other lesson, should possess unity and coherence. This unity may be obtained in various ways. (a) Problems in a list may be similar in the mathematical relations involved. (b) Problems may all relate to the same life situation, or grow out of a project. (c) They may be chosen because they require, or admit of, some particular method of thinking (as the analytic method), or some special type of objective aid (as the line diagram). (d) The teacher will often select problems that illustrate the use of some particular method of solution or manner of treatment, as the formula method, the equation method, tabular treatment, graphic treatment, etc.

3. Rarely, and only for some good reason, should the teacher assign for home work problems that are difficult or unusual.

4. It is well in assigning problems for home work to have the problems read and to make sure that the language is understood; but the method of solution should not be discussed except in cases of special difficulty.

5. If there has been home work assigned, a few minutes of the recitation period should be taken for checking up the work and straightening out difficulties. Frequently it is well to give one problem as a test to determine the genuineness of the home work.

6. If it is planned to devote the recitation period to problem work, any of the following plans may be followed: (a) Give a list of problems to solve as seat work and say, "Let us see how many we can solve in ten minutes." At the close of the time have the problems explained and check up individual work. (b) Same as (a) except omit the explanations. (c) Have each child in turn tell how he would solve a problem. If the numbers are easy, he may compute mentally and announce the exact answer; if the numbers are unsuited to mental computation, he may compute with "rounded" numbers and announce an approximate answer. (d) Set a long list of problems and allow each to select the most interesting ones to solve. After a suitable time the individual work is stopped and the children by common consent decide which problems are worth explanation and discussion. (e) An occasional period may be given to a problem-solving "match." Each side has been collecting problems which they now announce for the other side to solve. The side which

gets the most correct answers in the time allowed wins the match. If a pupil sets a problem he cannot solve, his side is penalized 3 points. (f) There should be occasional exercises in solving in which the numerical values of the given terms are omitted. Such work should include only such relations as have previously occurred. This type of work serves to generalize upon previous experience, and paves the way for solution by formula and equation.

CONTENT AND ORGANIZATION OF PROBLEM WORK

Instructional uses of problems. — The real purpose of the study of arithmetic is a practical one — the solution of problems; this is literally the alpha and omega of arithmetic work. Often a new process is taught as a better way to get the answer to a problem (p. 97), and always a new process is applied in a variety of problems. Even the drill in the new technique of a process will often be given meaning — will have its motive — by the realization that the desired skill will help in solving problems. It should be noted, however, that interest in problems, as such, is very slight in young children. This interest, if nurtured by sound teaching, will steadily increase as the child advances. This suggests the need for carefully considering the content of problems.

Content must be appropriate to the grade. — In the early grades the problems must relate to the personal experience of the children for whom they are intended. The home, the school, and the neighborhood constitute the young child's world. The work in arithmetic should enable the pupils to think about their environment, their experiences, and their activities quantitatively. As time

goes on the child's world expands. His school studies in literature, science, history, art, etc., are liberating him from the narrow, selfish concerns of early childhood. There comes a time when he can be interested in current events, when the bulletin board with its clippings from newspapers and magazines is a center of interest. The problem work in arithmetic should keep pace with this development. Most modern textbooks give evidence that their authors are aware that problems should not only be *practical* but should be *practical for the children who are to solve them*. Others fall short. For example, in one extensive problem list there are 53 problems on thrift for the third grade, 10 for the fourth grade, 1 for the fifth grade, and none at all on this important topic in the higher grades. Obviously this was not intended to be so.

Problems of daily living. — During recent years several investigations have been made of the practical uses of mathematics in everyday life. In 1911, G. M. Wilson, then Superintendent of Schools in Connersville, Indiana, worked out a course of study in arithmetic the content and organization of which was strongly influenced by the needs and activities of the community. In a later study he collected 14,584 problems which had actually occurred in the lives of 4068 adult men and women during a period of two weeks. These people lived in the middle west in communities ranging in size from a hamlet with a population of 75 to a city with a population of over 78,000. The fact that the average number of problems per person is only 3.64 would indicate that arithmetic is very infrequently used; but in one town (population 150) the average was 1.4 problems while in another town (population

198) the average was 30.5 problems, or about 2.5 problems per business day. These numbers lead one to doubt that the reports as a whole were anywhere near complete; for why should the people in one town have over 20 times as much practical use for arithmetic as the people in another town of similar size? It is significant, however, that about 75 per cent of all the problems were concerned with buying, about 10 per cent with selling, and another 10 per cent had to do with money apart from buying and selling. Wilson's inference that ". . . arithmetic is used by most people only when something is bought or sold, or when it is necessary to measure or determine a matter directly connected with business" might have to be modified somewhat if he were to get really complete reports.¹

It is safe to say that the bulk of arithmetical problems has to do with money and trade, and an inspection of the textbooks in common use, particularly those for the grades below the seventh, shows that their authors are aware of the demands of trade. On the other hand it is gratifying to note the tendency to cater to interests and needs other than those that center about the mere getting of a living. We find in increasing numbers problems that connect with children's play, with their constructive activities, with their group interests in the home and in the school, with their duties and privileges as young citizens, and with their interest in solving problems just for fun. These are problems of daily living just as truly as are problems of trade, for it is just as necessary to *live*

¹ WILSON, G. M. — "A Survey of the Social and Business Usage of Arithmetic"; *Teachers College Contributions to Education*, 1919.

as it is to *make a living*. And since the numerical relations involved in non-commercial problems are present also in commercial problems there is no danger of loss in letting the interests of children largely determine the content of the problems they solve.

The present excellent tendency is to vitalize problem work by giving the socialized or real-situation problem. Such may occur in connection with the working out of some project, or as isolated problems. A *real* problem has a social quality if it occurs in the experience of the one who solves it, or if the situation in which it occurs can be brought home to the imagination of the pupil.

If, however, problems are selected for no other reason than the practical reality of their content, they are very likely to be lacking in mathematical range. It should not be forgotten that problems must be so selected as to afford needed practice in dealing with quantitative relations and in accurate habits of thought. Unless this principle of selection is observed, the practical problem is in danger of becoming like Disraeli's practical man, "one who practices the errors of his forefathers." Over-vitalization may demathematize as much as over-mathematizing may devitalize.

Problems with social content. — In the higher grades of the elementary school the children are ready for training in desirable social attitudes. The school should inculcate respect for all kinds of labor that ministers to human needs. The teacher who is himself keenly aware of the mutual character of human activity — who is appreciative of the value of the services of those who operate our railroads, dig our coal, supply us with food, clothing, and shelter,

minister to our amusement needs, give us beautiful things to enjoy in literature, music, and art — will find in such activities a wealth of problem material. Newspapers and magazines are full of data rich in human interest; the pamphlets and reports issued by various social organizations and welfare agencies contain much interesting material; government bulletins are often rich in suggestions for problem material. If the teacher is broadly interested in human affairs, he can create in his pupils attitudes toward life that will tend to make them more conscientious in their vocations, more intelligent as citizens and more sympathetic and helpful in all the relations that they will be called upon to sustain. The work in arithmetic can be made to contribute to these desirable ends. We shall now give some illustrations of such problem material.

Cost of education. — Here are a few short lessons about the cost of schools.

1. Despite an increase since 1920, the figures indicate that school expenses are still less, proportionally, than other public costs. In the following table are shown the per cent of national income expended for schools and for all governmental activities from 1913 to 1921:

YEAR	PER CENT EXPENDED FOR ALL GOVERNMENTAL PURPOSES	PER CENT EXPENDED FOR PUBLIC SCHOOLS
1913	9.24	1.51
1914	10.18	1.67
1915	9.90	1.68
1916	7.85	1.41
1917	9.26	1.30
1918	19.74	1.25
1919	28.24	1.35
1920	14.17	1.48
1921	16.73	2.13

From the New York *Evening Mail*, Feb. 13, 1923.

Find the ratio of school expenditures to all governmental expenses for each year, and place these ratios in a fourth column. Has the proportion of school expenses increased or not? What caused the high governmental expenses since 1918? How did the schools fare during the war?

2. In Cincinnati, Ohio, the Lafayette Bloom Junior High School erected in 1914-15 cost 19.13 cents per cubic foot. Compare this with 45.5 cents per cubic foot for the elementary and junior high school building in the same city erected in 1923-

. . . Forest Park High School, planned to accommodate 2000 pupils at a cost of \$1,124,200, was erected at the unit cost of 27.6 cents per cubic foot. (Baltimore Bulletin of Education.) How many cubic feet of space is allowed for each pupil? Find the cost per pupil of this building.

A series of problems could be worked out about the cost of a year's schooling for each pupil in the class. Education reports are easily available, and the data can be treated simply enough to be understood by children in the upper grades. The solution of this problem gives opportunity for teaching proper care of school property.

"Our Men in Gray." — The following is a digest of an article by Katherine Mayo which appeared in the "Weekly News" published by the New York League of Women Voters. It illustrates a kind of problem that is abundantly available.

OUR MEN IN GRAY

In 1917 the State of New York established a State police force. These "men in gray" protect the lives and property of the people outside the large cities just as our men in blue protect the lives and property of the people of New York City.

At the present time, Nov. 1923, the State police force consists of 384 officers and men, organized into six troops, each troop being assigned to take care of a definite section of the State. The Troopers' motto is the Golden Rule; their courtesy, fearlessness, helpfulness, and sense

of honor and responsibility have caused the people to appreciate their services and to love and respect them as good friends.

During the year 1922 the Troopers recovered and returned to the owners \$983,213.60 worth of property that had been stolen; they also collected fines amounting to \$158,603.10. One-third of all the arrests made outside Greater New York were made by the Troopers, and 91 per cent of these arrests resulted in convictions. They also made 18,109 investigations of troubles of various kinds which, through their tact and wisdom, were settled without making arrests. The fire insurance companies estimate that the work of the Troopers has decreased the losses of farm buildings by fire to the extent of half a million dollars per year.

In his work of patrolling the territory assigned to him the Trooper commonly uses a motorcycle, but there are parts of the state in which the roads are so bad and there are seasons in the year when the snow is so deep that the motorcycle cannot be used; then the Trooper must travel on horseback. During the year 1922 the total distance traveled by all these men was 1,838,572 miles. When we consider that the fine service of these men cost the people of New York State only \$946,280.85 for the year mentioned, and that the population of the State is about 11 millions, it is clear that we are getting a great deal for the few cents it costs each one of us.

QUESTIONS

1. How many years since our state established its police force?
2. What is the average size of a troop of state police?
3. The area of New York State is 49,204 square miles. What is the average area under the protection of a single troop? In charge of a single Trooper?
4. In 1922 Greater New York with an area of 315 square miles was patrolled by 10,870 police. What was the area in charge of each man, on the average?
5. The population of New York City in 1922 was about 5,620,000. There was one policeman for how many people?
6. Find the ratio between the population outside the city and the number of Troopers.

7. On the average, how many miles did a Trooper travel per day in 1922?

8. Did the Troopers save the state enough money to pay for their own support? How much was clear money gain? Don't forget the reduction in fire losses.

9. What benefits did the people of the state gain which cannot be measured in dollars and cents?

Problems with vocational guidance value. — There comes the time in the lives of many elementary-school children when they must face the question of choosing a vocation. It is possible for the teacher to devote some time to problems grouped about the various vocations offered by the local community. It is a mistake for the teacher to place the "white collar jobs" in too attractive a light; every kind of useful work should be thought of as desirable. The question which the child should be led to answer is not, "Which job is cleanest, or nicest, or most high toned?" It is rather, "At which job can I succeed? What kind of work will give me the most satisfaction?"

The children can be encouraged to visit places where work is going on and to collect all kinds of information about the character of the work, what training is needed, what wages are paid, whether there is steady employment, whether one has to be absent a great deal from home, etc. A lesson, or series of lessons, could be given to the arithmetical side of the above questions, and textbook problems pertaining to the vocation being studied could be hunted out by the children and solved.

Organization as determined by arithmetical content. — In the primary grades the dominant aim is to master the mechanics of the fundamental operations. Accordingly

the problem work must be selected and organized so as to supply the motive for, and give meaning to, a new process, and to meet the need for interesting practice material. Problems are used for these purposes in all the grades, but because of the steady decrease in the amount of new work in abstract processes there is a corresponding decrease in this use of problems as we advance in the grades. In selecting problems for the purposes under discussion the teacher should avoid those that involve unfamiliar relations. The problems should be easy to understand and interesting in content. Whether they are of practical utility is of relatively little importance.

Problems growing out of projects. — While project problems afford valuable practice in the mechanical processes this practice is a by-product rather than the chief end sought. The purpose of project problems is to convince pupils of the reality and utility of arithmetic as a means of meeting practical human needs.¹

Most textbooks of recent date make frequent use of problems grouped about some interest or activity of children — the *play* store, the class picnic, the support of a baseball team, etc. Such textbook projects have high value in themselves, and will often suggest to the teacher ways of utilizing the immediate experiences of the children under her care.

Problems growing out of situations found in children's reading. — This type of problem work is illustrated on

¹ We are here using the term *project* in its narrow sense. Broadly speaking, the term is used to include not merely things of strictly utilitarian value but also things that awaken interest and curiosity. We are not concerned with defining *project*, but in embodying the spirit of the project method in the problem work.

page 291. The alert teacher will find frequent opportunity to break down the walls of partition that needlessly separate the child's school work into school subjects. Stories the children read often lead to construction work, to plan drawing, to mapping, etc. Many topics in geography, history, civics, and current events demand, or at least admit of, arithmetical interpretation.¹

Like every good thing the project, or real situation, problem can be misused and overdone. The teacher who possesses broad culture and who keeps well informed about worth-while things and events of real importance is the one best fitted to use such problem material, but such a teacher must guard against the danger of over-emphasis.

EXERCISES

1. Formulate a problem in addition suitable to the first year and show how a child would solve it by counting objects.

2. Treat similarly a subtraction problem of the take-away type; one of the addition type (p. 50).

3. Formulate a suitable problem in multiplication and show how a child would solve it (a) before he knows how to add, (b) when he knows how to add, but cannot multiply.

4. Formulate a *quotition*, or measurement, problem and solve it (a) by counting objects, (b) by addition, (c) by subtraction. For what grade is each method suitable?

5. Mention some of the characteristics of a good problem for each of the primary grades. Collect or make some problems that possess one or more of these characteristics.

6. The interest of a problem depends in part upon the way in which it is stated. Illustrate by stating the same problem in various ways.

¹ For suggestions concerning problem material, see: Thorndike, E. L., — *The Psychology of Arithmetic*, pp. 266–277; Macmillan. Overman, J. R. — *Methods and Principles of Teaching Arithmetic*, pp. 242–252; Lyons and Carnahan.

7. How can the teacher make the children's oral explanations of problems natural and spontaneous?

8. Show how you would help a pupil who adds instead of subtracts in one of the subtraction problems in Exercise 1, page 267.

9. State the difficulties that children may encounter in solving simple problems and discuss, or illustrate, how each kind of difficulty might be overcome.

10. Discuss the habits that function usefully in solving simple problems.

11. Find, or make, one problem of each type listed on page 262. Check the problems which you think are suited to children of the fourth grade. Select a committee from your class; let this committee collect the individual lists and make a list of good problems classified (a) according to type, (b) according to content or interest appeal.

12. Select from the above list, or other source, five two-step problems suitable for an initial lesson on complex problems, and explain in detail how you would help children with the solutions by the method of synthesis. In which of these problems would some objective aid be helpful? In which ones would *acting* the problem be helpful?

13. Explain in detail how the method of analysis might be used in the above lesson.¹

14. Show how you would use the method of analysis, with the diagram device, in helping children who are unable to solve problems 4, 5, 6, 7, pages 267, 268.

15. Make, or find, a list of problems well suited to illustrate the principles of unity stated on page 278.

16. If the problem in Exercise 7, page 268, were assigned as home work, what points in the language might require explanation? Explain them as you would to a class.

17. Conduct a class exercise in problems (a) according to the method suggested in 6 (c) on page 279; (b) according to the method in 6 (f).

18. (a) Take a page of miscellaneous problems in some textbook; select therefrom a suitable homework assignment, and mention the points you would take up with the pupils by way of preparing them

¹ The ideal procedure in such exercises as these is to plan and rehearse the teaching methods and then try them out with a group of children who have had no experience with such problems.

to do the work intelligently. (b) Explain how you would conduct the recitation on this assignment. Note: It is important to decide at the outset for what grade the work is intended.

19. In the *New York Times* of July 13, 1924, there was an article on the value of the Panama Canal as measured by the increasing use made of it. The following table shows by years the number of passages made through the canal and the *tolls* collected. Look up the original cost of the canal, the amount of the yearly payments made on account of our taking over the Canal Zone, and any other data of interest. Make a series of problems suitable to an eighth-year class.

Here is the story of the canal, told in a table published in *The Canal Record*, the official journal of the Canal Zone:

	TRANSITS	TOLLS
1915	1075	\$4,367,550
1916	758	2,408,089
1917	1803	5,627,463
1918	2069	6,438,853
1919	2021	6,172,828
1920	2478	8,513,935
1921	2892	11,276,889
1922	2736	11,197,832
1923	3967	17,508,199

20. The following was clipped from the *Christian Science Monitor* of May 22, 1923. It would make interesting material for use in geography, history, and arithmetic. Work up this story in its arithmetical aspects, comparing this method of communication with the present air service of the Post Office Department.

THE PONY EXPRESS

The Pony Express, that band of brave and determined men, and almost as determined horses, which kept the Pacific coast linked with the western end of telegraphic communications in the Mississippi Valley for about 18 months three-quarters of a century ago, is to be revived again this year. Beginning at its original starting point, in St. Joseph, Mo., the Pony Express riders, 200 strong, will carry the mail once more from the Father of Waters to the Father

of Oceans, more than 2000 miles. The ride will start before day-break on the morning of Aug. 28, and will finish in San Francisco, on the afternoon of Sept. 9, in the midst of the celebration of Admission Day by the entire State of California.

The first Pony Express rider left St. Joseph, Mo., at 4 A.M., April 3, 1860, according to H. C. Peterson, of the historical research department of the California State Library. In this service of the sixties, 420 horses, all the small California mustang, selected because of its speed and endurance, were used, and there were 80 riders, of whom Col. William F. Cody ("Buffalo Bill") was probably the best known, having been placed in charge of a division. About 400 men also were stationed at the posts along the route to care for the horses, there having been 190 of these stations. These stations were 15 and often 25 miles apart, and the length of the division covered by one man varied from eight hours riding through level country, where there were few Indians and plenty of water, to only four hours in mountainous country. Sometimes the men at the stations were killed and the relief horses run off by marauding Indians, and then the rider had to keep on going. There are records of one man and one horse having covered more than 300 miles in such a time of necessity.

The telegraph wire ended at St. Joseph, and the Pony Express connected that western wire terminus with the thousands of lately come immigrants to California, but its tenure of life was shortened by the offering of a prize of \$40,000 by the United States Government in 1861 for the extension of the telegraph line to Salt Lake City. Edward Creighton performed the feat, the California Telegraph Company carried the line on to San Francisco and the Pony Express became one of the romantic ghosts of the old days of the far west. A New York and a San Francisco newspaper once exchanged editions *via* the Pony Express by printing each edition on tissue paper, on which most letters were written, since the first fee for letters over the Pony Express was \$5.00 per half-ounce. This was later reduced to \$2.50.

Carrying Lincoln's Message

The best ride made over the old trail was seven days and 17 hours, when President Lincoln's inaugural address was delivered. Johnny Fry, believed to have been the first rider out of St. Louis at the opening of the Pony Express, made the best time of any rider in the carrying of the Lincoln message. He rode from St. Joseph to Seneca, 80 miles, and won the prize of \$5000 offered for the best time made by any rider. His time seems to have been lost; at least, the writer can find no record of it.

21. The following is quoted from a news item.¹ Make a series of problems based upon this material.

Washington, June 13 — A report just issued by Roy A. Haynes, Federal Prohibition Commissioner, indicates that during the fiscal year ended June 30, 1923, a low record was made in the withdrawals of whisky, the amount having been 1,754,893 gallons, the major part of which was dispensed by druggists on physicians' prescriptions. During the fiscal year 1921, the first full year of prohibition, withdrawals amounted to 8,671,860 gallons. There has been a steady diminution since that year.

It is pointed out that the high point of whisky consumption in this country was reached in the fiscal year of 1917, when there was a tax payment on 164,291,294 gallons of distilled spirits. The average annual consumption during the ten years prior to prohibition was about 130,000,000 gallons, present consumption of whisky officially released on permits being only a little more than 1 per cent of that prior to prohibition.

¹ *Christian Science Monitor*, June 13, 1924.

CHAPTER XI

GRAPHS

TEACHER'S KNOWLEDGE

Meaning and uses. — In the chapter on problems we have discovered that the relations involved in certain



SHRINKAGE OF CHINESE POPULATION IN THE UNITED STATES

In addition, there were 23,507 in Hawaii in 1920, as compared with 21,674 in 1910;
and 43,805 in the Philippines in 1918, as compared with 41,035 in 1903

FIGURE 39

situations are more easily recognized when graphic aids are employed to picture these relations.

The graph is a vivid and convenient way of stating the facts and is particularly helpful in situations which involve large numbers, or which contain data that can easily be tabulated. The graph may be called the illustrated story form of acquainting the mind with facts or relations which

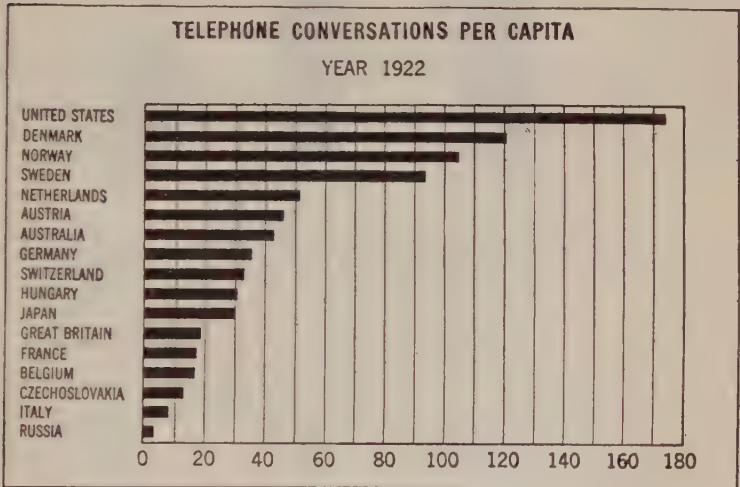


FIGURE 40

expressed in ordinary form would make very little appeal and would utterly fail to disclose their true significance.

Many different forms of graphs are in use, but the underlying principles of construction are similar in all. We shall now consider briefly some of the more important forms.

The picture graph. — The picture in Figure 39 shows the shrinkage of Chinese population in the United States in a period of ten years. The scale may easily be deter-

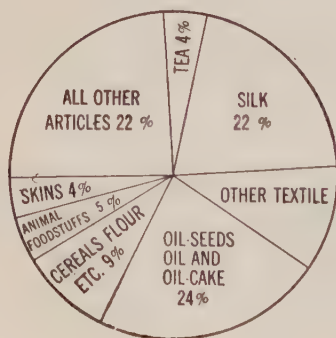
IS EVERY CHILD ENTITLED TO A COMPETENT TEACHER ?



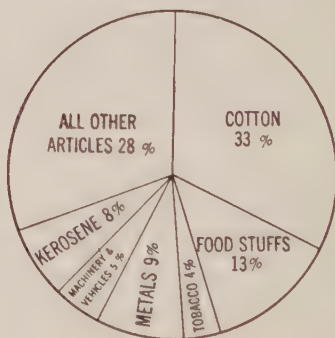
REASONABLE SALARIES ARE NECESSARY
TO OBTAIN COMPETENT TEACHERS

Division of Research
National Education Association

FIGURE 41



WHAT CHINA SOLD IN 1919



WHAT CHINA BOUGHT IN 1919

FIGURE 42

mined by finding the number of Chinese represented by one inch of height of any one of the given figures.

The bar graph. — A bar of any convenient width is used in either horizontal or vertical position and the length of each bar selected to represent any given item in the

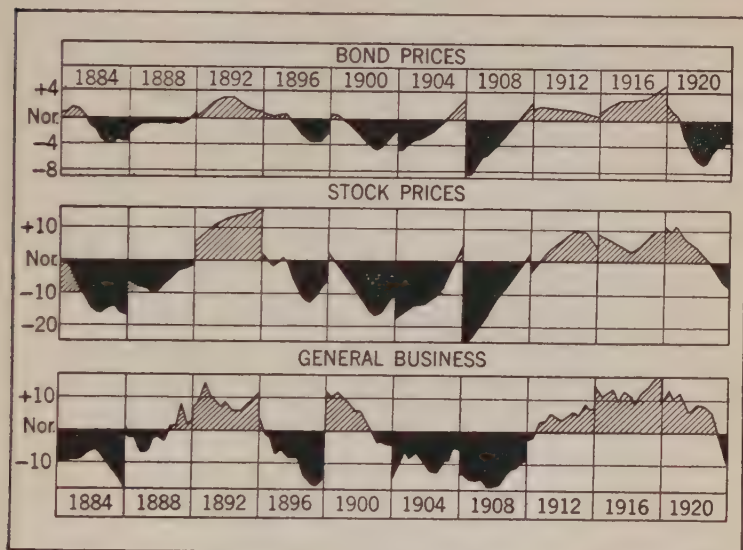


FIGURE 43

table of statistics is drawn to scale. The horizontal bar graph on page 295 shows the arrangement usually employed in this form of graph. The scale employed is ten to each space, as indicated.

Figure 41 illustrates a vertical bar graph.¹ The scale used here is indicated at the left.

¹ *The Journal of the National Education Association*; November, 1924.

The circle graph. — This is a form of graph in which relative values are vividly portrayed. Figure 42 shows certain facts about China's foreign trade.



FIGURE 44

Shaded areas. — In this form of graph the distribution is pictured by the size or density of shaded areas. The graph on page 297 pictures the deviation from normal in prices of stocks and bonds, and in general business, during a series of election years.

Another variation of this type of graph is shown in Figure 44.

Line graphs of related variables. — In this form of graph two scales are employed, one to represent the facts given and the other to represent some other item upon which these facts depend. Its purpose is to depict fluctua-

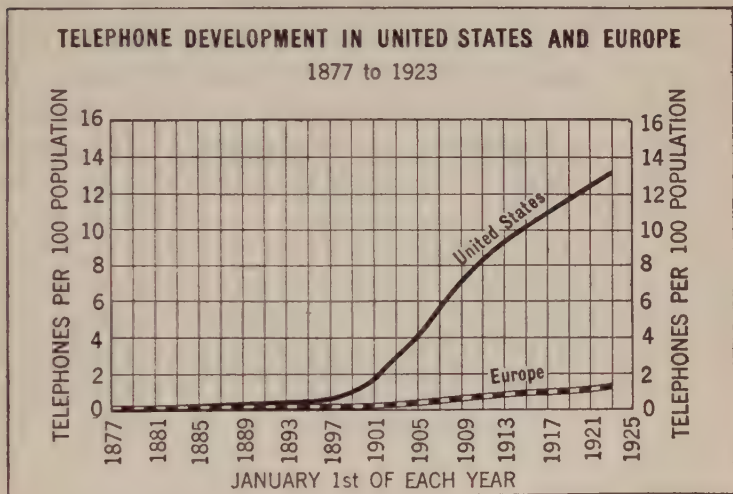


FIGURE 45

tuations of various kinds. Increases and decreases together with rates of change are easily read. To read a line graph find the item desired in either the horizontal or vertical scale and run up or across the line passing through this point until it intersects the graph; the reading in the other scale opposite this intersection is the desired information. The above line graph shows the telephone development in the United States and Europe for a period of forty-six years.

To determine the number of telephones per 100 in any one year, as 1909, follow the vertical line that passes through 1909 to the graph; then follow along the horizontal at this point to the scale at the right. The point lies midway between 6 and 8. Hence the number is 7. The increase in the number of telephones between the years 1877 and 1899 is very gradual and small. Since 1901 the increase in the United States has been very rapid. The contrast between the growth of the telephone business in the United States and in Europe is also vividly depicted.

Graphs in educational measurements. — The graph is a valuable means of interpreting statistical facts in the field of education. Otis¹ says it is a “method of interpreting statistics — one’s own interesting data collected painstakingly for an important purpose.” He maintains that we may speak of a *science* of educational measurement because “The method of handling the data of mental measurement statistically in order to determine how inexact they are and what degree of reliance may be put upon them constitutes one of the most exact of sciences.”

The graphic forms most commonly used in educational measurements are the bar graph, the line graph, and the correlation graph. Let us study concrete illustrations of these forms.

The following² is an ingenious combination of frequency table and bar graph, the bars being made of stars on the typewriter.

¹ OTIS, ARTHUR S. — *Statistical Method in Educational Measurement*, p. 1; World Book Company.

² STENQUIST, JOHN L., and WEIDMAN, MARY E. — “Results of Standard Tests Showing the Need for More Flexible Organization in Our Schools”; *Baltimore Bulletin of Education*, February, 1925.

Graphic Distribution Chart Showing Variation in Ability to Comprehend by 609 Pupils in the 6th Grade. Monroe Silent Reading Test.

(Each star represents the position of 2 pupils, each dot the position of 1 pupil)

	Score	Freq.
NORMAL GRADE ABILITY	0	1 .
	1	
	2	
3d grade 3.8 →	3	
	4	
	5	8 ****
	6	20 *****
4th grade 7.7 →	7	38 *****
	8	56 *****
5th grade 9.8 →	9	76 *****
	10	84 *****
6th grade 11.0 →	11	77 *****.
7th grade 12.5 →	12	71 *****.
8th grade 13.7 →	13	59 *****.
	14	37 *****.
	15	30 *****
	16	16 *****
	17	21 *****.
	18	7 ***.
	19	5 **.
	20	2 *.
	21	1 .

Range : (See Frequency Column Above)

Total Number 609

Median for 609 pupils all in 6th grade is 11.2

Note. — This chart shows that these 6th-grade comprehension scores range from 0 to 21, or from below normal 3d-grade ability to above normal 8th-grade ability.

By "normal grade ability" is meant the *median score* for the country at large for children of the grade indicated. The *range* is from 0 to 21. The median score in this group is the score of the 305th pupil, there being 304 pupils who rank below him and 304 above him. There are 283 pupils whose scores are below 11, hence the 305th pupil in the whole group is the 22d pupil in the group that scored 11. The abilities of these 22 pupils are assumed to range from 11 to $11\frac{7}{7}$, which would give the 22d pupil a score of $11\frac{21}{7}$, or about 11.2.

The *average* is found by dividing the sum of the individual scores by the number of individuals.

Since the *median* score is, in this distribution, in the middle of the *range* the median is practically the same as the average. This means that the distribution closely approximates the *normal probability curve*, a bell-shaped curve which is symmetrical with respect to a median line. A curve which departs from the normal form is either *skewed* or is irregular.

The *mode*, often taken as a *central tendency*, is the most frequent score; in the illustration the mode is 10.¹

There are several ways of expressing the degree of *variability* in a distribution, but it is beyond the scope of this chapter to discuss these methods. A very easily computed measure of variability is the "semi-interquartile-range," often denoted by *Q*. In the illustration the score one-fourth the way down from the poorest is 153d in the series, or 30th in the group with score 9. This score is about 9.4. The score three-quarters of the way down

¹ For a concise treatment of statistical methods, see Thurstone, L. L., *Fundamentals of Statistics*, The Macmillan Company.

from the poorest (or one quarter up from the best) is 25th in the group with score 13. This score is 13.4.

$$Q = \frac{13.4 - 9.4}{2} = \frac{4}{2} = 2$$

It is evident that the more widely the scores deviate from the central tendency the larger will be the value of Q .

The following is a good illustration of the line graph, taken from the article cited above. The dotted line

Graph showing median score and norms by grades in Monroe Arithmetic Test.

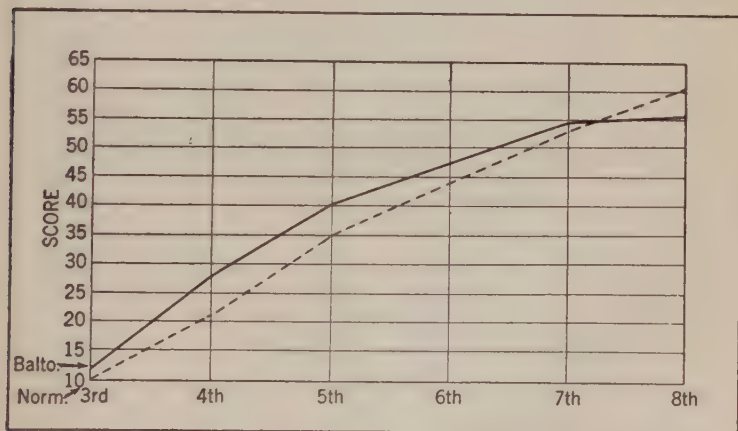


FIGURE 46

shows how the median ability of children throughout the country improves from the third to the eighth grade. The continuous line shows that the median abilities of Baltimore children is above the norm for all grades except the eighth.

The following graph shows the degree of *correlation* between the judgment of the teacher about his class of 40 children and the scores made by these children in a

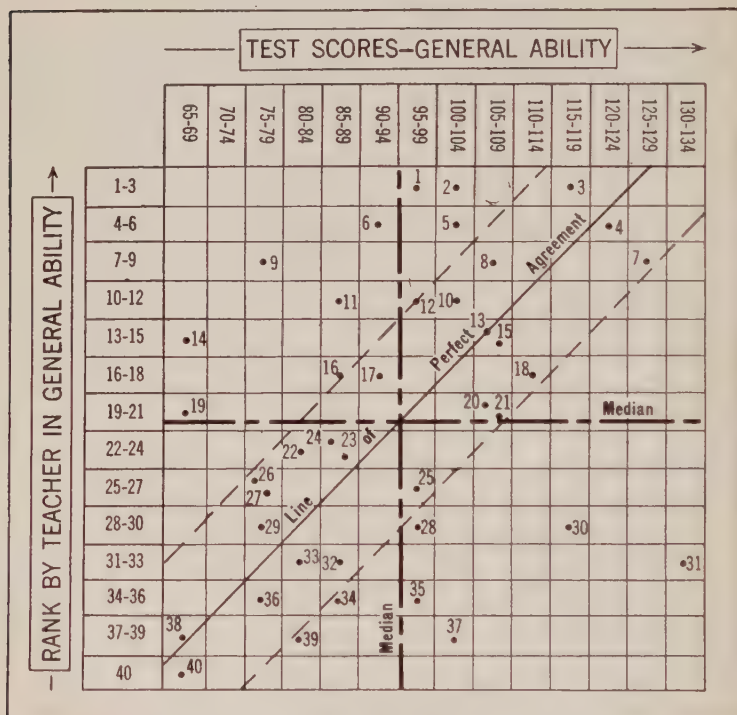


FIGURE 47

Chart showing relation between rank by teacher and test scores in general ability.

group of tests. Each dot stands for a pupil, the number over the dot being used in lieu of the child's name. If the teacher's judgment and the test scores agreed exactly

— that is, if each child ranked the same in both respects — all the dots would be on the *line of perfect agreement*. The fact that 24 of the dots come between the outer parallel lines indicates a fairly close agreement, or fairly high correlation, between teacher's judgment and test scores.

EXERCISES

TELEPHONE CONVERSATIONS AND TELEGRAMS IN
IMPORTANT COUNTRIES FOR THE YEAR 1922

COUNTRY	WIRE COMMUNICATIONS PER CAPITA		
	Telephone Conversations	Telegrams	Total
Australia	42.6	3.0	45.6
Austria	45.8	1.3	47.1
Belgium	16.6	0.7	17.3
Czechoslovakia	12.9	0.4	13.3
Denmark	120.3	0.7	121.0
France	17.0	1.7	18.7
Germany	35.2	1.0	36.2
Gt. Britain and No. Ireland	18.2	1.4	19.6
Hungary	30.5	0.8	31.3
Italy	7.7	0.5	8.2
Japan	29.7	1.1	30.8
Netherlands	51.3	0.8	52.1
Norway	104.5	1.6	106.1
Russia	3.1	0.2	3.3
Sweden	93.4	0.7	94.1
Switzerland	32.7	0.8	33.5
United States	174.0	1.7	175.7

1. Construct bar graphs to show (a) the number of telephone communications per capita for the countries listed; (b) the number of telegrams per capita. Explain how the data in these two graphs might be combined in one graph.

CAN AMERICA AFFORD EDUCATION¹ ?

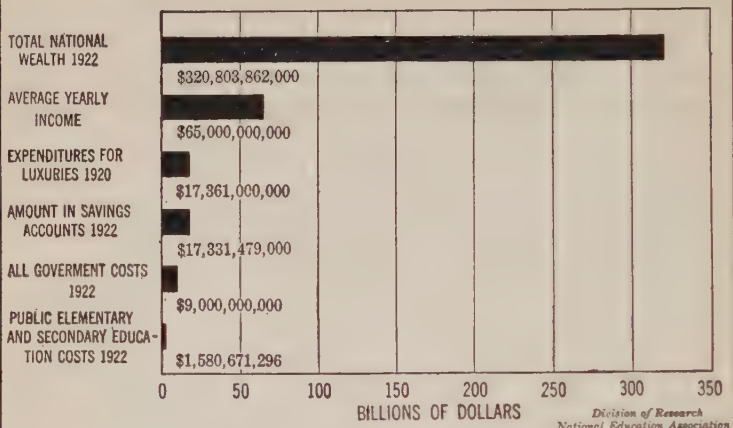
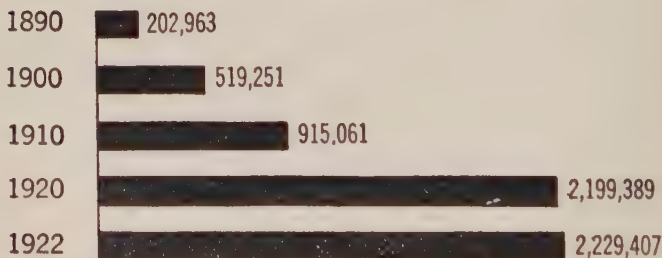


FIGURE 48

NUMBER OF STUDENTS ENROLLED IN PUBLIC HIGH SCHOOLS



IF THE POPULATION OF THE UNITED STATES HAD INCREASED AS RAPIDLY AS ITS HIGH SCHOOL ENROLLMENT SINCE 1890 ITS GENERAL POPULATION WOULD NOW BE 687,861,581

FIGURE 49

2. Determine from the graph (Figure 48) the relation of the cost of education in the United States to its average yearly income. What per cent of the total national wealth is the average income? How many times as much does the United States spend for luxuries as for education?

3. (a) Determine the scale used in the graph (Figure 49). (b) Make up five questions that may be answered from the data given.

4. Read and explain the bar graph (Figure 50) and tell where and how

you would look for the items from which the graph was made. Verify the construction of the graph.

5. In the picture graph (Figure 51) are the relations expressed by the heights of the bottles, by the areas, or by their volumes? In other words, which of these equations is true as between 1921 and 1923:

$$\frac{1.8}{3.2} = \frac{h}{h_1}, \quad \frac{1.8}{3.2} = \frac{h^2}{h_1^2}, \quad \text{or} \quad \frac{1.8}{3.2} = \frac{h^3}{h_1^3}?$$

6. Using a protractor for measuring the circle graph (Figure 52), determine the relation of the number of telephones in the United States to the number (a) in Great Britain; (b) in Germany; (c) in France; (d) in all other European countries. (e) Determine from the figures given the relation of the number of telephones in Europe to the number in the United States.

7. Verify the statement made below the graph in Figure 53.

8. (a) Interpret the graph (Figure 54). (b) During what year was the rate of increase slowest? Most rapid?

9. Construct a line graph to show the increase in weight as related to height for (a) a ten-year-old girl; (b) a ten-year-old boy; (c) a sixteen-year-old girl; (d) a sixteen-year-old boy. (See pages 310 and 311.)

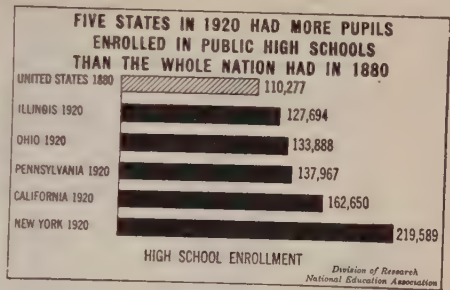


FIGURE 50

¹ *Journal of the National Education Association*, November, 1924.

THE MOUNTING DEATH-RATE FROM ALCOHOLISM

Deaths from alcoholism in 1920 were 1 per 100,000 population; in 1921, 1.8; in 1922, 2.6; and in 1923, 3.2.



FIGURE 51

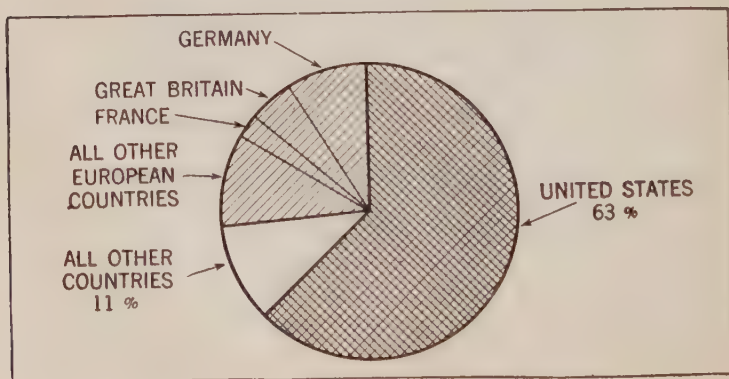
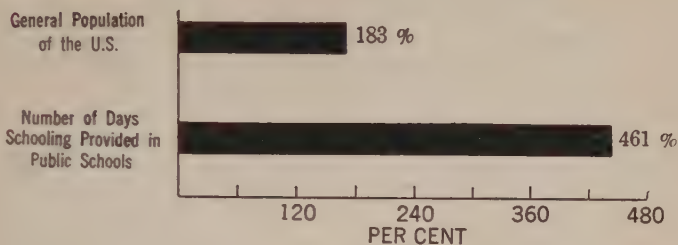


FIGURE 52

SCHOOL ATTENDANCE HAS INCREASED MORE RAPIDLY THAN GENERAL POPULATION

PER CENT INCREASE 1870 TO 1922



If the population of the United States had increased as rapidly as school attendance, the population in 1922 would have been 216,227,633 instead of 109,248,393

Division of Research
National Education Association

FIGURE 53

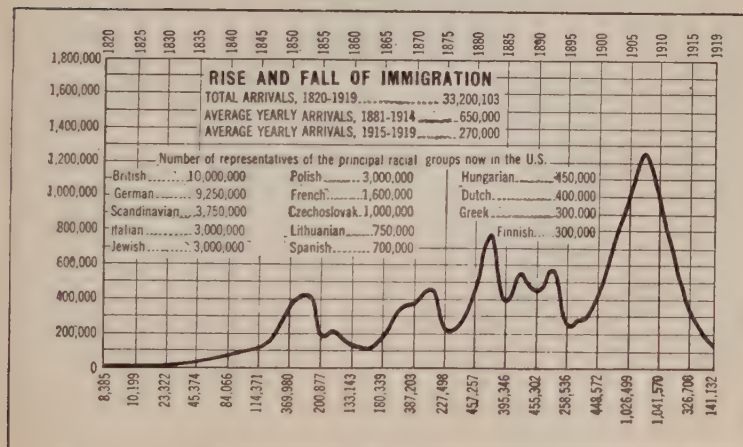


FIGURE 54

HEIGHT AND WEIGHT TABLE FOR BOYS

HEIGHT, INCHES	5 YRS.	6 YRS.	7 YRS.	8 YRS.	9 YRS.	10 YRS.	11 YRS.	12 YRS.	13 YRS.	14 YRS.	15 YRS.	16 YRS.	17 YRS.	18 YRS.
39	35	36	37											
40	37	38	39											
41	39	40	41											
42	41	42	43	44										
43	43	44	45	46										
44	45	46	46	47										
45	47	47	48	48	49									
46	48	49	50	50	51									
47		51	52	52	52	54								
48		53	54	55	55	56	57							
49		55	56	57	58	58	59							
50			58	59	60	60	61	62						
51			60	61	62	63	64	65						
52			62	63	64	65	67	68						
53				66	67	68	69	70	71					
54				69	70	71	72	73	74					
55					73	74	75	76	77	78				
56					77	78	79	80	81	82				
57						81	82	83	84	85	86			
58						84	85	86	87	88	90	91		
59						87	88	89	90	92	94	96	97	
60						91	92	93	94	97	99	101	102	
61							95	97	99	102	104	106	108	110
62							100	102	104	106	109	111	113	116
63							105	107	109	111	114	115	117	119
64								113	115	117	118	119	120	122
65									120	122	123	124	125	126
66									125	126	127	128	129	130
67									130	131	132	133	134	135
68									134	135	136	137	138	139
69									138	139	140	141	142	143
70										142	144	145	146	147
71										147	149	150	151	152
72										152	154	155	156	157
73										157	159	160	161	162
74										162	164	165	166	167
75											169	170	171	172
76											174	175	176	177

HEIGHT AND WEIGHT TABLE FOR GIRLS

HEIGHT, INCHES	5 YRS.	6 YRS.	7 YRS.	8 YRS.	9 YRS.	10 YRS.	11 YRS.	12 YRS.	13 YRS.	14 YRS.	15 YRS.	16 YRS.	17 YRS.	18 YRS.
39	34	35	36											
40	36	37	38											
41	38	39	40											
42	40	41	42	43										
43	42	42	43	44										
44	44	45	45	46										
45	46	47	47	48	49									
46	48	48	49	50	51									
47		49	50	51	52	53								
48		51	52	53	54	55	56							
49		53	54	55	56	57	58							
50			56	57	58	59	60	61						
51			59	60	61	62	63	64						
52			62	63	64	65	66	67						
53				66	67	68	68	69	70					
54				68	69	70	71	72	73					
55					72	73	74	75	76	77				
56					76	77	78	79	80	81				
57						81	82	83	84	85	86			
58						85	86	87	88	89	90	91		
59						89	90	91	93	94	95	96	98	
60							94	95	97	99	100	102	104	106
61							99	101	102	104	106	108	109	111
62							104	106	107	109	111	113	114	115
63							109	111	112	113	115	117	118	119
64								115	117	118	119	120	121	122
65								117	119	120	122	123	124	125
66								119	121	122	124	126	127	128
67									124	126	127	128	129	130
68									126	128	130	132	133	134
69									129	131	133	135	136	137
70										134	136	138	139	140
71										138	140	142	143	144
72											145	147	148	149

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10. The telephone rate from Englewood, N. J., to Danbury, Conn., is 35¢ for any time up to 3 minutes and 10¢ for each additional minute or fraction thereof. (a) Make an equation to express the cost of a call between these points for any number of minutes, using C for cost and E for extra time in minutes. (b) Make a line graph from which may be read the cost for any number of minutes.

11. $C = \frac{\pi}{2} d$ is the formula for circumference of a circle in terms of the diameter. Draw a line graph to show the same relation and read the value of C for $d = 7$, $d = 14$, $d = 21$, $d = 4$, $d = 10$.

12. The following records were made in a test in column addition given to two classes, 4 A and 3 B.

4 A : 60, 50, 80, 90, 45, 65, 95, 100, 40, 80, 70, 75, 80, 85, 70, 65, 90, 80, 60, 65, 85, 45, 55, 80.

3 B : 50, 80, 40, 35, 100, 40, 80, 70, 60, 65, 90, 60, 65, 90, 60, 55, 50, 65, 70, 75, 80, 45, 65, 70, 65, 65.

(a) Construct a graph for 4 A ; for 3 B.

(b) From the graph determine the median, mode, and range of each class.

(c) Find the average for each class.

(d) Find the overlapping of the classes.

METHODS OF TEACHING

Incidental teaching. — The use of graphs may be introduced early in the course, but their systematic study should not be attempted below the seventh year. The teacher can start the work by posting a chart for the class attendance record. Each morning he plots the attendance, making such remarks as will attract attention to the fact and the method of recording it. When the children are ready they may assume the duty of keeping the record. In this informal way other graphs can be made — a temperature graph, a graph of performance in arithmetic or other subject, graphs showing health data, etc.

Systematic treatment. — There is a pronounced tendency to teach graphs systematically in the seventh, eighth, and ninth grades. It is customary to begin with the picture graph and follow with the bar graph, the circle graph, and the line graph. Scale drawings and maps showing density of population, rainfall, etc., are sometimes introduced. Many texts also include floor plans of houses and other scale drawings.

In general it is well to give practice in the interpretation of a particular kind of graph before requiring the pupils to construct a graph. Also, it is probably easier for a pupil to make a graph from data presented in tabular form than from the same data in its crude state. It is possible and helpful to have the children tabulate the results of school tests and then represent them by either a bar or line graph.

Graphs as aids in motivating abstract drill. — In the intermediate and advanced grades the children are old enough to understand simple graphs. Any graphic device that conveys to the child definite information as to his status — his abilities as compared with those of his classmates and with accepted norms — will serve to encourage him to do his best. He can know where he stands, in what things he needs most practice, and will feel a responsibility for his own improvement. The teacher should know how to use the elementary methods set forth in this chapter for the purposes here stated, and also as an aid in making reports to his official superiors.

EXERCISES

1. Make a list of school activities or interests that could be utilized for the incidental work with graphs. For each item in your list state what type of graph is appropriate, and prepare a form for the graph.
2. Assuming that the pupils know how to use the protractor, draw up specific directions for them to follow in making a *circle* graph to show how the school day should be apportioned for the purposes of eating, sleeping, home study, school, play, helping in the home.
3. Plan a first lesson in interpreting line graphs; in making line graphs.
4. Describe some ways in which graphs may be used in motivating the work in the fundamental operations in the upper grades.
5. Devise a graph form by which a pupil could keep track of his improvement in abstract computation. How would it be used?

CHAPTER XII

DENOMINATE NUMBERS

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions. — We have seen in Chapter I that the early felt need of expressing *how much* gave rise to the invention of various units for measuring a quantity. In answer to the question *How many?* the people of the world evolved a plan of counting beautiful in its simplicity and its symmetry. In strong contrast, the means of answering the question *How much?* have been complex and inconvenient to a degree. Instead of a constant base, as we have in the decimal system of counting, the tables for measuring are as haphazard as it is possible to imagine.

To answer the question *How much?* the name of the unit must be expressed, but in answering the question *How many?* the name of the unit is not expressed. This gives rise to *concrete* and *abstract* numbers. An abstract number is a number in which the name of the unit is not expressed, as 6; a concrete number is a number in which the name of the unit is expressed, as 6 houses. If the unit named has been standardized by law or custom, the concrete number is called a *denominate number*, as 6 miles. A denominate number may have the name of but one unit or of more than one unit in the same table, giving

rise to the *simple denominate number*, as 6 feet, and the *compound denominate number* (called in the older arithmetics compound number), as 6 ft. 8 in.

Indefinite units. — The first units of measure were crude and indefinite. Just as the plan of counting was the outgrowth of using the most convenient means at hand — the fingers of the hands — so the units of measure were evolved from natural units convenient to employ. The movements of the heavenly bodies furnished an easy way of reckoning time: the time from sunrise to sunrise was the basis for the day; the time between a phase of the moon and its recurrence, for a month; and the time that it took the sun to pass through successive changes from one position in the heavens to the same position was the basis for a year. In measuring distance, dimensions of land were measured by steps and longer distances by a day's journey. In measuring weight, grains of wheat or barley were used, and in measuring capacity, bowls and drinking vessels were employed. Units of value were not needed, as barter was the means of exchange.

Definite units. — As civilization advanced the need arose for more definite units. These were adopted as convenience prompted and were afterwards standardized by law or custom, so that gradually there arose well-defined units for measuring lengths, surfaces, volumes, weight, capacity, value, time, and arcs of circles. As stated above, these various measures arose in haphazard manner in response to practical needs. Hence the development of the subject matter in a teacher's study of measures is historical rather than logical. One system of weights and measures is strictly logical in its construction.

This is the metric system which in recent years has spread widely and has practically supplanted the older measures in almost all the countries of the world, except the English-speaking nations. In the following paragraphs a brief description is given of the English tables and of the metric system.

Measures of length. — The primitive method of measuring length by steps gave way to units based upon parts of the human body, as the foot. Other such units were: the distance from the joint to the tip of the thumb, later standardized as a twelfth (*uncia*) part of the foot or *inch*; the length of the forearm, or cubit; the girth of the body or length of the outstretched arm, the *yard* (possibly from *gyrdon*, girth, or possibly *gerda*, stick). The length of the *rod* was fixed by sixteen men standing in line so that the toe of each one's left foot touched the heel of the man in front. The *mile* was originally a thousand *paces* (*milia passuum*).

Measures of surface. — For measuring surfaces, the units are based upon the linear units: thus, the *square inch*, a square each side of which is an inch; the *square foot*, *square yard*, *square rod*, and *square mile*, squares each side of which is respectively a foot, a yard, a rod, and a mile. A unit larger than the square rod and smaller than the square mile was found convenient for measuring land; hence an extra unit, the *acre* (*aecer*, field), was introduced.

Measures of volume. — The units of volume are also based on the linear units as follows: *cubic inch*, *cubic foot*, and *cubic yard*, cubes whose edges are respectively an inch, a foot, and a yard.

Measures of weight. — In measuring weight, *grains* (of barley or wheat) were used. 7000 of these made the *pound* (*pendus*, a weight) by which heavy articles of low value were weighed. It was called the Avoirdupois pound (*aver*, goods; *de*, of; *peis*, weight). Goods of greater value were weighed by a lighter pound of 5760 grains called the Troy (from the city of Troyes) pound; a twelfth (*uncia*) of this pound was called an *ounce*; the weight of 24 grains was a *pennyweight*. The lighter pound of 5760 grains was also used for weighing medicines and known as the Apothecaries' pound; 20 grains made a *scruple* (*scrupulum*, little weight); 3 scruples made a handful or *dram* (*drachma*, hand); 8 drams made an *ounce* or twelfth part of a pound.

Measures of capacity. — For measuring liquids we have: the *pint* (*pinta*, a mark, supposed to be the marked part of a larger vessel); the *gallon* (perhaps from *jale*, a bowl); the *quart*, or fourth part of a gallon (*quartus*, fourth); and the fourth part of a pint or *gill* (*gelle*, a drinking vessel for wine). For measuring dry commodities, a different sized quart and pint, and the larger units *peck* (a quantity picked up), and *bushel* (*bussula*, a little box) are used.

Circular measure. — In circular or angular measurement we have a division into 360 parts (probably based upon the old Babylonian year of 360 days) called *degrees* (*de*, down; *gradus*, step). A degree was divided into 60 equal parts called *minutes* (*minutem*, a little part) and each minute into 60 equal parts called *seconds* ($\frac{1}{60}$ part, or second small part of a degree). It is thought that this sexagesimal scale (p. 156) was devised because of its

convenience, relating, as it does, the measurement of time with the measurement of a circle.

Time measure. — The more definite units of time were, like the primitive ones, based upon the movements of the heavenly bodies. The *day*, or the length of time for the rotation of the earth on its axis; the *month* (a moon), or length of time for the revolution of the moon around the earth; the *year*, or length of time for one revolution of the earth around the sun. For convenience the day was divided into *hours* (*hora*, a period, or time); the hour into *minutes* (*minutem*, a small part); and the minutes into *seconds* (*secundus*, a second small part). Days were grouped into *weeks*, a day being assigned to each of the seven planets in the order of their supposed distances from the earth, which, according to the Romans, was as follows: Sun, Moon, Mars (Germanic Tyr), Mercury (Woden), Jupiter (Thor), Venus (Freya), and Saturn.

Measures of value. — Each country has its own units of value. Our system of money is based upon the French. The basic unit, the *dollar*, is divided into the *dime* (*disme*, tenth), *cent* (*centime*, hundredth), and *mill* (*mille*, thousand).

The Metric system. — The units used differed greatly among different peoples, and until the eighteenth century no one attempted to improve this hodge podge. At this time the French invented a plan of measuring as beautiful and perfect as the Hindu plan of counting. This system, known as the Metric system, was adopted by them in 1799 and is now in use in almost every civilized country in the world.

The following graph shows the rapidity of its advance. Strange to say, the two great English-speaking nations,

despite their acknowledged leadership in world affairs, have been slow in adopting it. Although it was legalized in the United States in 1866, it has not come into general use except in the scientific world. Perhaps if it had been invented before instead of after the American Revolution, we might have adopted it as we did the French plan of notation and our table of money, patterned after the French. Were this system

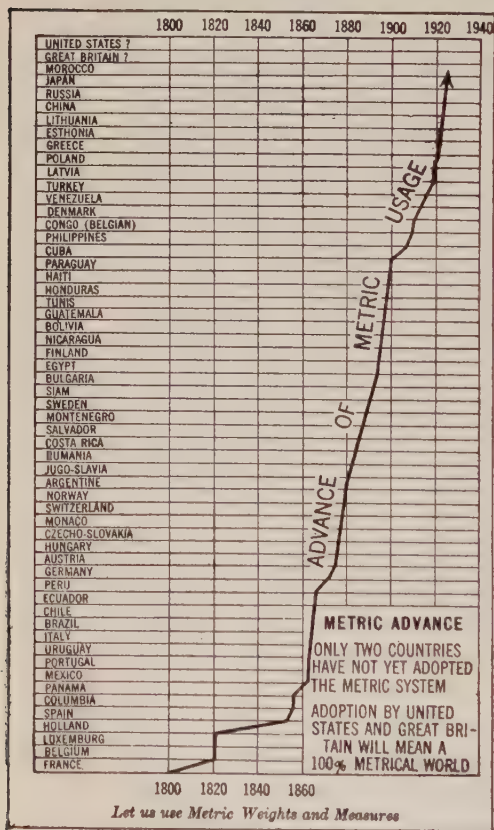


FIGURE 55

tables of measure would be as rhythmic and pleasurable to learn as is counting.

In the *Metric System* units were adopted for measuring

length, capacity, weight, wood, and land. Each unit is subdivided into tenths, or *deci*-units; hundredths or *centi*-units; and thousandths or *milli*-units. Multiples of each unit are also powers of ten, — ten times a unit is called a *deka*-unit, a hundred times, a *hekto*, a thousand times, a *kilo*, and ten thousand times, a *myria*.

The meter, or unit of length, was standardized as one ten-millionth of the distance from the equator to the pole. The other four units are based upon the meter; the unit of capacity is a cubic decimeter and called a *liter*; the unit of weight is the weight of a cubic centimeter of water and called a *gram*; of land measure, a dekameter square and called an *are*; of wood measure, a meter cube and called a *stere*. The units for measuring surfaces other than land, and volumes other than wood, are based upon the units of length as in the English tables, thus: *square meter*, *cubic meter*.

English tables in common use. — The following tables are in general use:

Length	Dry Measure
12 inches (in.) = 1 foot (ft.)	2 pints = 1 quart (qt.)
3 feet = 1 yard (yd.)	8 quarts = 1 peck (pk.)
$5\frac{1}{2}$ yards = 1 rod (rd.)	4 pecks = 1 bushel (bu.)
320 rods = 1 mile (mi.)	
Square Measure	Avoirdupois Weight
144 sq. in. = 1 sq. ft.	16 ounces (oz.) = 1 pound (lb.)
9 sq. ft. = 1 sq. yd.	100 pounds = 1 hundred-
$30\frac{1}{4}$ sq. yd. = 1 sq. rd.	weight (cwt.)
160 sq. rd. = 1 acre (A.)	20 cwt. = 1 ton (T.)
640 acres = 1 sq. mi.	

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Circular, or Arc Measure

60 seconds (") = 1 minute (')
 60 minutes = 1 degree (°)
 360 degrees = 1 circumference

U. S. Money

10 mills = 1 cent (ct. or ¢)
 10 cents = 1 dime
 10 dimes = 1 dollar (\$)
 10 dollars = 1 eagle

Cubic Measure

1728 cu. in. = 1 cu. ft.
 27 cu. ft. = 1 cu. yd.
 128 cu. ft. = 1 cord (cd.)

Time

60 seconds (sec.) = 1 minute (min.)
 60 minutes = 1 hour (hr.)
 24 hours = 1 day (da.)
 7 days = 1 week (wk.)
 4 weeks = 1 month (mo.)
 $\left\{ \begin{array}{l} 12 \text{ months} \\ 52 \text{ weeks} \\ 365 \text{ days} \end{array} \right\} = 1 \text{ year (yr.)}$
 366 days = 1 leap year
 100 years = 1 century

Liquid Measure

4 gills (gi.) = 1 pint (pt.)
 2 pints = 1 quart (qt.)
 4 quarts = 1 gallon (gal.)

Counting

12 things = 1 dozen (doz.)
 12 dozen = 1 gross
 12 gross = 1 great gross

Paper

24 sheets = 1 quire
 20 quires = 1 ream
 2 reams = 1 bundle

Equivalents

1 gal. = 231 cu. in.
 1 bu. = 2150.4 cu. in.
 1 cu. ft. water weighs $62\frac{1}{2}$ lb.
 $7\frac{1}{2}$ gal. = 1 cu. ft.
 1 lb. Av. = 7000 grains
 1 lb. Troy = 5760 grains
 1 lb. Apoth. = 5760 grains
 £ 1 = \$4.8665
 1 franc = \$.193
 1 lira = \$.193
 1 mark = \$.238

The number of days in a month varies :

Thirty days hath September,
 April, June, and November ;
 All the rest have thirty-one,
 Except the second month alone,
 To which we twenty-eight assign
 Till leap year brings us twenty-nine.

All years divisible by 4, except century years, are leap years ; all century years divisible by 400 are leap years.

Other English tables.—The following tables are in less common use :

Surveyors' Long Measure

7.92 in. = 1 link (li.)	
$\left\{ \begin{array}{l} 100 \text{ links} \\ 66 \text{ ft.} \\ 4 \text{ rods} \end{array} \right\} = 1 \text{ chain (ch.)}$	
80 chains = 1 mile (mi.)	

Troy Weight

24 grains (gr.) = 1 penny-weight (pwt.)	
20 pwts. = 1 ounce (oz.)	
12 ounces = 1 pound (lb.)	

English Money

4 farthings (far.) = 1 penny (d.)	
12 pence = 1 shilling (s.)	
20 shillings = 1 pound (£)	

Apothecaries' Fluid Measure

60 minims (m) = 1 fluid dram (ʒ)	
8 fluid drams = 1 fluid ounce (ʒ)	
16 fluid ounces = 1 pint (O.)	
8 pints = 1 gallon (Cong.)	

Surveyors' Square Measure

625 sq. li. = 1 sq. rd.	
$\left\{ \begin{array}{l} 16 \text{ sq. rd.} \\ 10,000 \text{ sq. li.} \end{array} \right\} = 1 \text{ sq. ch.}$	
10 sq. ch. = 1 A.	
640 A. = 1 sq. mi. or section	
36 sq. mi. = 1 township	

Apothecaries' Weight

20 grains (gr.) = 1 scruple (ʒ)	
3 scruples = 1 dram (ʒ)	
8 drams = 1 ounce (ʒ)	
12 ounces = 1 pound (lb.)	

French Money

10 centimes = 1 decime	
10 decimes = 1 franc	

Writing Prescriptions

In writing prescriptions the Roman numerals are usually used and a final i is written as a j ; the abbreviations precede the number. Illustration: ʒij ʒiij ʒiij for 2 ounces, 3 drams, 2 scruples.

Miscellaneous

1 hand = 4 inches	1 knot, or
1 span = 9 inches	nautical mile = 6080 feet
1 fathom = 6 feet	1 league = 3 miles
1 furlong = 40 rods	1 perch(stone) = 24½ cu. ft.

Metric tables.—In general it is possible to construct any table in the Metric system from the name of the basic

unit and the prefixes. Thus, using the unit *meter* with the prefixes we get the table of linear measure (the principal units are in heavy faced type).

10 millimeters (mm.) = 1 centimeter (cm.)

10 cm. = 1 decimeter (dm.)

10 dm. = 1 meter (m.)

10 m. = 1 dekameter (Dm.)

10 Dm. = 1 hektometer (Hm.)

10 Hm. = 1 kilometer (Km.)

10 Km. = 1 myriameter (Mm.)

All the Metric tables are similarly constructed by adding the basic unit to the prefixes, except that in the table of weight we have two extra units: 10 myriagrams = 1 quintal; 10 quintals = 1 ton (metric). In land measure the units most frequently used are the decare, the are, and the hektare; in wood measure, the decistere and the stere.

Measurements of temperature. — The first attempts to measure temperature were made about the close of the sixteenth century. The invention of the thermometer is attributed to Galileo. In 1593 he made an open-air thermoscope consisting of a bulb with a long tube attached, which was provided with a scale and dipped below the surface of a colored liquid — water or wine. Some of the air was expelled from the bulb and so the liquid rose to different heights according to the changes in temperature, to which it was very sensitive. The invention of the type of thermometer used at the present time is also attributed to Galileo, about 1612. Alcohol was the first liquid used in such thermometers.

In 1661 Fabian made a scale in which he used as fixed temperatures those of snow and midsummer heat. In

1694 Renaldini proposed the freezing and boiling points of water as the fixed temperatures. Fahrenheit of Amsterdam introduced his alcohol thermometers about 1709 and his mercury ones about 1714. He introduced the present thermometer about 1720, although it is probable that he himself did not devise the instrument. It became generally known in Europe in the first half of the eighteenth century and is still in common use. The boiling point on the *Fahrenheit thermometer* is 212° above zero and the freezing point is 32° above.

In 1742 Celsius of Sweden introduced a centigrade scale. This scale was adopted by France at the time of the Revolution and is used in the *Centigrade thermometer*. The boiling point on this thermometer is 100° above zero and the freezing point is 0° . About 1731 Réaumur devised a different scale, in which the freezing point was 0° and the boiling point 80° . This *Réaumur thermometer* was until recently in extensive use on the continent of Europe.

Longitude and time. — *Definitions.* — Geographers and navigators find it convenient to locate places on the earth's surface by stating the distance east or west of a given line and north or south of another line. The reference line from which north and south distances are measured is the equator. Since there are 360 degrees in a circumference, the greatest distance north or south of the equator is 90 degrees, the distance of either pole. The north or south distance from the equator measured in degrees is called the *latitude* of a place. The circumferences of the small circles parallel to the circle whose circumference is the equator are called *parallels of latitude*. Any parallel is a direct east-and-west line.

Imaginary lines on the earth's surface joining the poles are called *meridians*. The meridian which passes through Greenwich, England, is called the *prime meridian*. The distance in degrees east or west of the prime meridian is called the *longitude* of a place. The greatest longitude a place can have is 180 degrees east or west.

As the earth rotates on its axis once every 24 hours, the sun appears to pass over 360° of longitude in 24 hours of time. Hence a difference of 15° in longitude between two places corresponds to a difference of 1 hour of time. As the sun appears to move in the heavens from east to west the time at a place 15° west of another is 1 hour earlier. Thus, it is 10 A.M. at a place 45° west longitude when it is 9 A.M. at a place 60° west longitude. If it is noon at the prime meridian, it is *post meridiem* (P.M.) or *afternoon* at places east of it; while it is *ante meridiem* (A.M.) or *forenoon* at places west of it.

The international date line. — For convenience, the 180th meridian has been selected to mark the beginning

of each day for the whole earth. This is called the *International Date Line*. Suppose it is 10 A. M. Monday at Greenwich; 180° east of

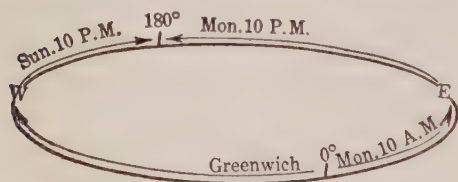


FIGURE 56

Greenwich it is 12 hours later or 10 P.M. Monday; 180° west of Greenwich it is 12 hours earlier or 10 P.M. Sunday. Hence in traveling east one travels at last over into west longitude and hence into the earlier day. That is, if it is Monday 10 P.M. before crossing the line

as he travels east, it will be Sunday 10 P.M. after crossing the line.

Standard Time.—For convenience in travel in the United States, the railroad companies in 1883 adopted a system of *Standard Time*, by which the United States is divided into four time belts each extending approximately $7\frac{1}{2}^{\circ}$ east and west of the 75th, 90th, 105th, and 120th meridians respectively. The boundary lines between the belts are irregular lines connecting important railroad centers. All places within any one belt have the same time and a difference of 1 hour exists between any belt and its adjacent belt. The change of time is made at the boundary line. The names of the time belts are the Eastern, Central, Mountain, and Pacific. Thus when it is 10 A.M. at all places in the Eastern Time Belt, it is 9 A.M. in the Central, 8 A.M. in the Mountain, and 7 A.M. in the Pacific.

Reductions. — Facility in reducing from one denomination to another is often needed as a foundation for problems in measurements. Modern textbooks are not devoting much space and attention to reductions. The reductions illustrated below are more difficult than would be needed in ordinary problems solved by the pupil. They are used because they illustrate *to the teacher* the technique of solution better than would those of ordinary difficulty.

Reductions from lower units to higher (sometimes called *reduction ascending*), or from higher to lower (*reduction descending*), are really problems in themselves and should be so treated. The following are illustrations of such reduction problems.

(a) Reduce 18,500 inches to *integers* of higher denominations.

$$18,500 \text{ in.} = ? \text{ int. higher denom.}$$

$$18,500 \text{ in.} = 1541 \text{ ft. 8 in.}$$

$$1541 \text{ ft.} = 513 \text{ yd. 2 ft.}$$

$$513 \text{ yd.} = 93 \text{ rd. } 1\frac{1}{2} \text{ yd.}$$

$$\frac{1}{2} \text{ yd.} = 1 \text{ ft. 6 in.}$$

$$18,500 \text{ in.} = 93 \text{ rd. 2 yd. 1 ft. 2 in.} \quad \text{Ans.}$$

$$5\frac{1}{2})513$$

$$11)1026$$

$$93, 1\frac{1}{2}$$

$$1 \text{ yd. 1 ft. 6 in.}$$

$$2 \text{ ft. 8 in.}$$

$$2 \text{ yd. 1 ft. 2 in.}$$

(b) Reduce 5 bu. 3 pk. 5 qt. 1 pt. to pints.

$$5 \text{ bu. 3 pk. 5 qt. 1 pt.} = ? \text{ pt.}$$

$$5 \text{ bu. 3 pk.} = 23 \text{ pk.}$$

$$23 \text{ pk. 5 qt.} = 189 \text{ qt.}$$

$$189 \text{ qt. 1 pt.} = 379 \text{ pt.} \quad \text{Ans.}$$

$$23$$

$$189$$

$$8$$

$$2$$

$$184$$

$$378$$

$$5$$

$$189$$

The operations. — The four operations explained below come later in the course. They are of infrequent occurrence in life practice, and will be interesting and valuable only as they are logically related to the corresponding operations with integers. To teach them mechanically or to drill upon them unduly would be a waste of time.

Addition. — We proceed as in addition of integers, adding the units of each column separately, reducing each sum *as we proceed* to the next higher order when the sum is equal to one or more of that order. An airplane flew 5 hr. 30 min. at one time, 6 hr. 48 min. at another and 8 hr. 15 min. at another; what was the total flying time?

1st time,	5 hr. 30 min.	1	<i>Explanation.</i> The sum of the minutes is 93 minutes or 1 hour 33 minutes. We write 33 in minutes' column and carry the 1 to the hour column. The sum of the hours is 20 hours. We write 20 in the hours' column.
2d time,	8 hr. 15 min.	60)93	
3d time,	6 hr. 48 min.	60	
Total,	?	33	
Total,	20 hr. 33 min.	Ans.	

For subtraction, multiplication, and division, the computation only is illustrated. Concrete situations may be supplied by the teacher.

Subtraction. — We proceed as in subtraction of integers, applying the principle of the Austrian method as in the following illustration. From 1 rd. 1 yd. 1 ft. take 3 yd. 2 ft.

$\begin{array}{r} 1 \text{ rd. } 1 \text{ yd. } 1 \text{ ft.} \\ \underline{\quad 3 \text{ yd. } 2 \text{ ft.} \quad} \\ 2\frac{1}{2} \text{ yd. } 2 \text{ ft.} \end{array}$ or 3 yd. 6 in., <i>Ans.</i>	<i>Explanation.</i> To 2 ft. enough must be added to give 1 ft. in feet's column, or 1 yd. 1 ft. or 4 ft. 2 ft. must be added. 2 ft. and 2 ft. gives more than 1 yd.; hence 1 yd. must be added to 3 yd., making 4 yd. To 4 yd. enough must be added to give 1 yd. in yards' column, etc.
---	--

Explanation by other methods is left to the reader.

Multiplication. — We proceed as in the multiplication of integers, multiplying the number in each column separately and reducing to the next higher column, as in addition. Illustration: Multiply 1 yd. 1 ft. 10 in. by 3.

$\begin{array}{r} 1 \text{ yd. } 1 \text{ ft. } 10 \text{ in.} \\ \quad \quad \quad 3 \\ \hline 4 \text{ yd. } 2 \text{ ft. } 6 \text{ in.} \end{array}$ <i>Ans.</i>	<i>Explanation.</i> 3 times 10 in. equals 30 in., or 2 ft. 6 in. We write the 6 in. in inches' column and carry 2 ft. to ft. column. 3 times 1 ft. is 3 ft.; 3 ft. plus 2 ft. equals 5 ft., or 1 yd. 2 ft., etc.
---	--

Partition. — We proceed as in long or short division, according to whether or not the divisor is greater than 12. Illustration: Divide 4 yd. 2 ft. 6 in. by 3.

$\begin{array}{r} 3 \overline{) 4 \text{ yd. } 2 \text{ ft. } 6 \text{ in.}} \\ \underline{1 \text{ yd. } 1 \text{ ft. } 10 \text{ in.}} \end{array}$ <i>Ans.</i>	<i>Explanation.</i> 4 yd. $\div$ 3 = 1 yd. with 1 yd. remaining. We write the 1 yd. in yards' column. 1 yd. or 3 ft. and 2 ft. are 5 ft. 5 ft. $\div$ 3 = 1 ft., etc.
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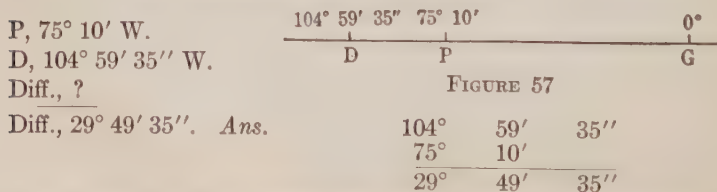
Quotition. — To find how many times one quantity contains another, we must reduce both to the same denomination and proceed as in ordinary division. Illustration: Divide 4 yd. 2 ft. 6 in. by 1 yd. 1 ft. 10 in.

$$\begin{array}{r}
 1 \text{ yd. } 1 \text{ ft. } 10 \text{ in.} = 58 \text{ in.} \quad 14 \\
 4 \text{ yd. } 2 \text{ ft. } 6 \text{ in.} = 174 \text{ in.} \quad 12 \\
 \hline
 3 \quad 168 \\
 58 \text{ in. } \overline{)174 \text{ in.}} \quad \text{Ans., } 3. \quad 6 \\
 \hline
 174
 \end{array}$$

Problems. — Problems involving the common units of measure and the units of the Metric system are solved as any ordinary problem. Nothing technical beyond a knowledge of the tables and ordinary reductions is necessary for success in solving them.

In longitude and time problems, graphic aids simplify the work. Mechanical aids, as memorizing statements like the following: "To find the difference in longitude between two places, subtract their longitudes if they have the same kind; if not, add," should not be used. Picture the relations, rationalize the situation, and find the results through an understanding of the relations. Illustrations:

(a) The longitude of Philadelphia is $75^{\circ} 10' \text{ W.}$; of Denver, $104^{\circ} 59' 35'' \text{ W.}$; what is their difference in longitude?



It is evident from the diagram that the difference in longitude will be found by subtracting.

(b) When it is Monday 10 A.M. at Albany, what time is it at Manila?

A, $73^{\circ} 44' 50''$ W.	$73^{\circ} 44' 50''$	0°	$121^{\circ} 30'$
M, $121^{\circ} 30'$ E.	A	G	M
A, Mon. 10 A.M.	FIGURE 58		
M, ?	840		
Diff. L, $195^{\circ} 14' 50''$	$15 \overline{)195^{\circ}}$	$14'$	$50''$
Diff. T, 13 hr. 59 sec.	13	0	59
Time M, 11:01 P.M. Mon.	Ans.	$15 \overline{)890}$	
		59	

What is the difference in longitude? (See a.) *Ans.*, $195^{\circ} 14' 50''$.
 If 15° of longitude corresponds to 1 hour of time, $15'$ to 1 minute of time, and $15''$ to 1 second of time, how many hours, minutes, and seconds in $195^{\circ} 14' 50''$? *Ans.*, 13 hr. 59 sec.

If it is Mon. 10 A.M. at Albany, what is the time at Manila where it is 13 hr. 59 sec. later? *Ans.*, Mon. 11:01 P.M.

(c) When it is 10 A.M. at Philadelphia, what is the time at Denver? Since Philadelphia is in the Eastern Time Belt and Denver in the Mountain Belt, there is a difference in time of 2 hours; it will be 2 hours earlier than 10 A.M. at Denver, or 8 A.M. *Ans.*

The graphic method of solving problems in thermometer readings is best. The relations may easily be discovered from the picture and the problem then solved rationally. If a person were solving thermometer problems continually, it would pay him to memorize the equivalents, but since the ordinary person meets them infrequently, he will find the rational method, aided by the graph, the more satisfactory. One or two illustrations will suffice to make the method clear.

(a) When the Fahrenheit thermometer registers 70° , what is the reading on the Centigrade and Réaumur? (See Figure 59.)

F, 70°	$\left. \begin{array}{l} \text{C, } 21\frac{1}{3}^{\circ} \\ \text{R, } 16\frac{2}{3}^{\circ} \end{array} \right\} \text{Ans.}$	$180^{\circ} \text{ F} = 100^{\circ} \text{ C} = 80^{\circ} \text{ R}$
C, ?		$1^{\circ} \text{ F} = \frac{5}{9}^{\circ} \text{ C} = \frac{4}{9}^{\circ} \text{ R}$
R, ?		$38^{\circ} \text{ F} = 21\frac{1}{3}^{\circ} \text{ C} = 16\frac{2}{3}^{\circ} \text{ R}$

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How many degrees between the freezing and the boiling points on the Fahrenheit thermometer? *Ans.*, 180° .

How many degrees above freezing is 70° Fahrenheit? *Ans.*, 38° .

If 180° Fahrenheit is equivalent

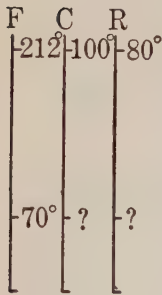


FIGURE 59

to 100° Centigrade and to 80° Réaumur, what is the equivalent of 1° Fahrenheit? *Ans.*, $\frac{5}{9}^{\circ}$ C and $\frac{4}{9}^{\circ}$ R.

What is the equivalent of 38° Fahrenheit? *Ans.*, $21\frac{1}{3}^{\circ}$ C and $16\frac{2}{3}^{\circ}$ R.

(b) When the Centigrade is 10° below zero, what is the reading of the other thermometers? (Fig. 60.)

C, 10° 100° C = 180° F = 80° R
R, ? 1° C = $\frac{9}{5}^{\circ}$ F = $\frac{4}{5}^{\circ}$ R
F, ? 10° C = 18° F = 8° R

R, 8° } *Ans.*
F, 14° }



FIGURE 60

If 100° Centigrade is equivalent to 180° F and 80° R, what is 1° Centigrade equivalent to? *Ans.*, $\frac{9}{5}^{\circ}$ F and $\frac{4}{5}^{\circ}$ R.

What is the equivalent of 10° C? *Ans.*, 18° F and 8° R. What is the reading of 18° below freezing on the Fahrenheit thermometer? *Ans.*, 14° F (above zero).

EXERCISES

1. Look up the derivation of *abstract* and *concrete*. Are these names suitably chosen?
2. Compute in miles the value of 1° of longitude at the equator.
3. Which is the largest fortune, £500,000, 10,000,000 francs, or \$2,450,000? Use equivalents on page 322.
4. Compare the above fortunes at current rates of exchange as quoted in the daily paper.
5. A man has one field 60 rods square; another, 35 rd. long and 18 rd. wide. What is his land worth at \$400 an acre?
6. At the cheese counter in the Manhattan market, two salesmen

were taking account of stock. One man weighed the pieces and the other put down the items as follows:

AMERICAN		SWISS		ROQUEFORT	
lb.	oz.	lb.	oz.	lb.	oz.
8	12	15	6	4	7
6	5	12	15	6	5
5	10	7	9	9	13
9	2	4	3	5	15
3	14	9	13	2	9
4	7	10	5	3	12

(a) Find the total amount of each kind of cheese.

(b) Find value of each kind if American is worth 32¢ a pound, Swiss 38¢ a pound, and Roquefort 45¢ a pound.

7. The height of five boys in the 2 A class was as follows: 4 ft. 6 in., 4 ft. 8 in., 3 ft. 10 in., 4 ft. 7 in., 3 ft. 11 in. Find the average height of these five boys.

8. Find the height of each boy in decimeters and the average height in decimeters.

9. A bin 10 ft. long, 8 ft. wide, and 5 ft. deep contains how many bushels?

10. A decimeter is 3.937 in. Find how many cubic inches a cubic decimeter or liter contains. What part of a gallon is it? What part of a bushel?

11. How many years, months, and days have elapsed since Lincoln was born? Since Columbus discovered America?

12. A boy went to bed at quarter of eight P.M. and got up at quarter past seven A.M. How long was he in bed?

13. Look up the dimensions of a freight car (marked on the outside of the car) and compute its volume. If filled with wheat to a depth of 5 feet, what would the wheat weigh? Compare your answer with the capacity in pounds marked on the car. (1 bu. wheat weighs 60 lb.)

14. Allowing the average size of each car in the train to be 50 ft. 6 in. long, 8 ft. 4 in. wide, and the wheat 5 ft. deep, how many cars would it take to hold the wheat crop of the United States in 1924?

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15. What would be the length of the train in miles, disregarding the space between the cars?

16. How long a distance would a million dollar bills cover if laid end to end?

17. By drawing to scale a square $5\frac{1}{2}$ yd. on each edge, show that $30\frac{1}{4}$ sq. yd. equals 1 sq. rod.

18. Discover the English equivalent (a) of 1 gram in grains; (b) of 1 Kg. in pounds; (c) of 1 are in acres; (d) of 1 stere in cords.

19. What date is 100 days after Christmas? 100 days before Christmas? See methods explained on page 449.

20. What is the value of a piece of ice 8'' by 10'' by 12'' at 70¢ a hundredweight? (1 cu. ft. of ice weighs 60 lb.)

21. How much water in gallons, quarts, and pints from the melting of 100 lb. of ice?

22. A telegram was sent from New York at 2 P.M. and received in San Francisco 2 hours later. What time was it at San Francisco when the message was received?

23. When it is 9 A.M. at Pekin, what is the time at London? at New York?

24. When it is 7 A.M. at longitude $56^{\circ} 30' W.$, what is the longitude where it is 9 P.M. of the same day?

25. What does the Centigrade read when the Fahrenheit is at "frost temperature" — 40 degrees?

26. A New York weather report for a day in July read *maximum 92 degrees, minimum 86 degrees*. What were the maximum and minimum Centigrade readings?

27. At what temperature Fahrenheit and Centigrade does alcohol freeze?

28. A man bought a tract of land 100 rods by 96 rods at \$1000 an acre and sold the entire tract for \$75,000. What did he gain?

29. Construct a graph showing the variation in temperature during any month for which you can secure a record of daily temperatures. (Use a World Almanac or other such reference.)

30. Bring in a recipe for a cake and compute the price of the articles needed for same. Ascertain the current prices at a grocery store.

SOME PUZZLES

1. A druggist buys 3 lb. of medicine by the Avoirdupois pound. He retails it in prescriptions averaging $3\text{v } \text{Dij gr.}\text{XII}$ each. How many prescriptions will it make?

2. At what time between 3 and 4 o'clock are the hands of a clock together?

3. How many prescriptions of $\text{℥iij } \text{℥iv } \text{Dij}$ can be put up from 4 lb. avoirdupois of quinine? How many 3 gr. capsules from the remainder?

4. Which is heavier, a pound of feathers or a pound of gold and how much?

5. Find what decimal part a quart of milk is of a quart of flour by finding the number of cubic inches in a liquid quart and in a dry quart. Find the answer true to 2 decimal places.

6. Compare by graphs the fortunes of a Frenchman worth 1,000,000 francs, an Englishman worth £50,000, and an Italian worth 400,000 lira (count rates of exchange at par).

7. Which weighs more, 100 gallons of water or the water contained in a tank 4 ft. long, 2 ft. wide, and 2 ft. deep?

METHODS OF TEACHING

Measurement in response to need. — In teaching denominate numbers the various units should be introduced as the child needs to use them, or as his comprehension enables him to appreciate their use. An interesting exercise in comparisons (*greater or less* judgments) consists in having the children arrange themselves in line in order of height; the expressions of judgment come spontaneously — as they must if the best results are to be obtained. Rough judgments of this sort lead naturally to the question of *how much* greater or less, which is the motive for measurement. In many classrooms measurements with indefinite units, or natural units, such as the natural foot, the hand-breadth, cubit (distance from the finger tip to the elbow), etc., are

omitted as unessential. When such units are used the difference in the numerical results obtained by different children shows the need for standard units.

Early steps in measurement. — In the primary grades the first exercises in measuring distance by a definite unit are with the foot, as it is easier for little fingers to manage the larger unit than to be accurate with the smaller inch. A little later the child may employ the inch in such exercises as constructing small paper boxes, paper chairs, paper beds for dolls, and making Christmas, Easter, and birthday cards, little booklets, etc. He also easily learns to measure with the quart and pint, and is familiar with the smaller coins—the cent, nickel, and dime. Exercises in easy problems to make him familiar with the exact values of these units should often be used.

In connection with the use of the foot and the inch in measuring lines, the making of preliminary estimates is of high value. Drawing by guess lines of stated length and then correcting with the ruler is the converse exercise.

Later on, the yard is of practical use. The mile should be made familiar to the city child by the number of blocks in a mile, and to the country child by the distance from one familiar place to another that measures about a mile. The time it takes to walk this distance helps to give an accurate idea of the mile. The peck and bushel can be taught by visits to the grocery store or to the farmer's granary; the gallon measure can be shown in the school and its relation to the quart found by finding how many quarts of water fill the gallon measure. The pound and ounce should be taught by the use of scales so that the children can actually weigh a pound, half-pound, and

quarter-pound of some dry commodity, as corn meal, oats, wheat, or salt. If it is not possible to obtain a scale, a pound bag of salt can be used and other commodities estimated by comparison with this.

Reductions from one denomination to another should be taught as soon as the relation between the units involved is known. This work should be given meaning through suitable problems.

Measurement of time. — The *day* may be introduced by reference to the time from sunrise to sunrise, or from sunset to sunset; later the fact that the legal day begins at midnight must be explained. The *month* may be taught by explaining the derivation of the name from *moon* — one month is approximately the time from one full moon to another. The days of the week, the number of weeks in a month, the number of months in a year and their names, should all be taught by reference to the calendar. The *hour*, *minute*, and *second* should be taught in connection with reading time from the clock.

In teaching time by the clock, it is important to decide in advance the order of procedure. The following is a good order:

- I. Telling the hours.
- II. Telling half past any hour.
- III. Telling quarter past any hour.
- IV. Telling the time in hours and minutes.
 - (a) The minutes being multiples of five.
 - (b) The minutes being any number.

The interest of the children can be aroused in class discussion of the time for their various daily activities: rising time, breakfast time, time to start for school, etc.

Under each heading in the above outline it is well to give two kinds of exercises — first telling what time the clock says ; second, making the clock say any specified time. A home-made toy clock with hands cut out of tin may be used.

Measurement of surfaces. — In teaching square measure the child should see a square foot divided into inch squares and find for himself the number of square inches in the square foot. A square yard may be drawn on the blackboard and measured by the square foot. Even the troublesome $30\frac{1}{4}$ square yards to the square rod¹ can be discovered by drawing to scale a square $5\frac{1}{2}$ units long by $5\frac{1}{2}$ units wide. The acre has to be introduced as a new unit, since it is not a square whose dimensions are a linear unit, but is equivalent to 10 squares, each of which is 4 rods long and 4 rods wide (the length of a surveyor's chain).

Cubic measure. — The table of cubic measure should be similarly derived from long measure by having the picture of a cubic foot showing the number of cubic inches in a layer and the number of such layers in the cube, from which the child can compute for himself the number of cubic inches in a cubic foot. A cubic yard drawn to scale and similarly subdivided into cubic feet should be used for deriving the next item in the table.

Learning the common tables. — It is questionable how much time should be spent in memorizing the tables of denominate numbers. Each table requires a great deal of time to fix in mind, for the English system of weights and measures is so heterogeneous a total as to cause dismay. The denominate-number field is a rich one for practical

¹ The number of square yards in a square rod is of no practical use and may be omitted if desired.

problem material, however, and time spent in memorizing the tables commonly used would be time saved in applying them in problem work. It is better that children should learn thoroughly the facts in the tables in common use, than learn more facts poorly. The place to master these useful tables is in grades four to six. The less-used tables have a certain interest as general information; they may be found in books of reference. Practice in using books of reference is in itself a valuable training, and is as necessary in arithmetic as in other subjects.

The Metric system. — In teaching the Metric system, the meter should be used and each child should have a small decimeter measure of his own.¹ By measuring short lengths with this, and by constructive exercises employing the decimeter and centimeter, he may become just as familiar with these units as with the foot and inch. For measuring by the *liter*, he should construct a pasteboard cube 1 dm. $\times$ 1 dm. $\times$ 1 dm., fill it with sawdust, and discover that it holds about a quart. A kilogram weight should be obtained if possible and its equivalent in pounds discovered by balancing the scales. Familiarity with the kilogram may be obtained by finding the weight in kilograms of each pupil in the class, perhaps making out a health chart for standard weight according to height and years, using the decimeter for recording heights and the kilogram for weights.

In solving problems in the Metric system, very little reduction to English denominate numbers should be

¹ Decimeter rulers and interesting information concerning the Metric system may be obtained from the *Metric Association*, New York City. See graph, page 320.

called for. Problems involving Metric units alone, to show the great superiority of that system, might help to



FIGURE 61

bring about the education needed in this country to force the Metric system into common use and to abolish the

tables that we now use, which, to say the least, are an anachronism in this age of system and efficiency.

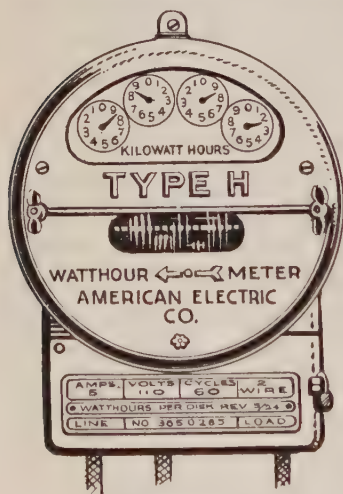


FIGURE 62

Problems and problem material. — There is an almost infinite variety of problems arising which involve measurements. Familiarity with the tables and with easy reductions is perhaps best secured by much concrete work. This is a rich field for mental problems and for problems in buying and selling. It also affords the most convenient

point of departure for teaching "household arithmetic" such as: (1) Reading gas, electric, and water meters and computing the monthly cost of the gas, water, and elec-

tricity consumed (see Figures 61, 62); (2) measuring extracts by the ounce and teaspoon; flour and sugar by the half-pint (teacup); weighing spices, tea, and condiments by the ounce, quarter pound, and half-pound; (3) reading thermometers; (4) computing the cost of meat, cheese, and other table necessities; (5) comparing in a rough way the cost of home-made bread with baker's bread; (6) studying food values of butter, milk, etc.¹

Problems in changing the size of recipes used in cooking will interest the girls. Estimating the cost of laying sidewalks, of paving streets, of building a playhouse, etc., are problems suitable for boys.

Bills and receipts. — In connection with a study of purchases for the home an excellent opportunity is presented for teaching the subject of *Bills and Receipts*. Children should bring to class various forms of bills and receipts. They should discover the essential items of a bill and perhaps make such a summary of these essentials as the following:

Various forms of bills are in common use. All show the name and address of the party selling the goods and of the purchaser; the date on which the bill is rendered; the quantity and price of the goods sold; and the terms of the bill. A statement of service rendered is also called a bill. When bills are paid a *receipt* is given. The customary form of receipt is to write the word *Paid*, the date of payment, and signature of payee. Sometimes a separate receipt is sent.

The following show some ordinary forms of bills:

¹ Much interesting material on food values of dairy products is put out by the *National Dairy Council*, Chicago, Ill., and by the *California Dairy Council*, San Francisco, Cal.

Telephone 1131J 6 OAKLAND ST., ENGLEWOOD, N. J., 9/22, 1923

Mr. J. APPLEQUEST.....Dodge "T"

To W. H. AHRENS, Dr.
Automotive Specialist
Repairs and Supplies

Terms Strictly Cash

2½ hr. labor @ 1.50

3 lb. Dixon's Graphite Cup Grease @ 40¢

3	75		
1	20	4	95

EXERCISES

- Write a series of questions about clock time which would be of practical interest to children of the second year.
- Write a series of exercises for each of the following:
 - A lesson on measurement with the foot rule, the inch not to be used.
 - A lesson on measurement with the foot rule, the inch to be used.
 - A lesson on the quart and pint.
 - A lesson on the mile.
- Make a list of practical problems suitable for oral work for third-grade children involving such units of measure as the yard, foot, inch, quart, pint, cent, nickel, dime, including easy reductions.
- Plan a lesson in constructive work, children to be taught how to make some article calling for measurement by the inch.
- Using a globe, explain the subject of longitude and time as to an upper-grade class.
- Plan a lesson (a) on teaching Bills; (b) on teaching Receipts.
- Plan a lesson on reading water meters.
- Using a gas bill, plan a lesson in teaching how to compute the cost of gas for a family.
- Write a series of problems for use of the ounce, quarter-pound, and half-pound in some phase of "household arithmetic."

H. C. F. KOCH & CO. INC.

West 125th Street

Dry Goods, House Furnishings, Furniture, Carpets, Etc.

Folio 341

NEW YORK,

June 1, 1924

Sold to

MRS. B. C. DAVIS,

5 Middlemay Circle,

Forest Hills, L. I.

Bills are due when rendered and prompt settlement is required

May									
9	1 Beads				1	00			
12	1 Bowl			39					
	1 Block			29					
	4 yds. Scrim	39	1	56					
	3½ yds. Satin	1 59	5	57	7	81			
19	1 Gloves		1	00					
	1 Spread		2	29					
	1 Coat		7	95					
	5 yds. Cretonne	39	1	95					
	5 yds. Cretonne	39	1	95					
	1 Nest Bowls			65					
	5 Paper			95					
	1 Cream			39	17	13			
27	1 Coat "RTND"				25	94		7	95
								7	95
									17 99

10. Show how to teach some interesting measures using recipes for cakes as motivation.

11. Show how the following material may be used for classroom work :

COMPONENT PARTS OF MILK

ACCORDING TO U. S. DEPARTMENT OF AGRICULTURE
FARMERS' BULLETIN No. 363

	WATER, PER CENT	PROTEIN, PER CENT	FAT, PER CENT	CARBOHY- DRATES, PER CENT	ASH, PER CENT
Whole milk	87.0	3.3	4.0	5.0	0.7
Skim milk	90.5	3.4	.3	5.1	.7
Cream (light)	74.0	2.5	18.5	4.5	.5
Buttermilk	91.0	3.0	.5	4.8	.7
Whey	93.0	1.0	.3	5.0	.7

The following table, compiled by specialists of the Department of Agriculture, shows the quantities of various foods needed to supply as much protein or energy as one quart of milk :

PROTEIN	ENERGY
One quart of milk is equal to : 7 ounces of sirloin steak ; 6 ounces of round steak ; 4.3 eggs ; 8.6 ounces of fowl.	One quart of milk is equal to : 11.3 ounces of sirloin steak ; 14.9 ounces of round steak ; 9 eggs ; 14.5 ounces of fowl.

Another method of comparison used by the Department of Agriculture shows that, as an economical source of protein, milk at 15 cents a quart is equivalent to sirloin steak at 34.9 cents a pound, or eggs at 37.7 cents a dozen.

As a source of energy, milk at 15 cents a quart is worth more than sirloin steak at 21.3 cents a pound, or eggs at 19.8 cents a dozen.

"A quart of milk a day for every growing child — is a good rule and easy to remember." — *Dr. Henry C. Sherman, Cornell University.*

CHAPTER XIII

MEASUREMENT OF GEOMETRIC FIGURES

TEACHER'S KNOWLEDGE

DEVELOPMENT OF DEFINITIONS

Meaning of mensuration. — In the preceding chapters we have discussed the different units of measurement which the peoples of the world have invented as they felt the need for them. Some of these measures have to do with *space* relations.

Our experience in space gives us concrete notions of one dimension, two dimensions, and three dimensions. It is the purpose of this chapter to define the more important geometric figures, to explain how they are measured, and to discuss methods of teaching the simpler of them.

One-dimensional terms. — That which has one dimension is called a *line*. A line may constantly extend in the same direction, may constantly change its direction, or may change its direction at intervals, giving rise to the *straight line*, the *curved line*, and the *broken line*.

Two lines in the same plane may or may not meet ; if they never meet, both are extending constantly in the same direction and the lines are said to be *parallel*. If they meet, the lines are extending in different directions and the difference in direction is known as an *angle*. When two straight lines meet in such a way that the adja-

cent angles thus formed are equal, the two lines are said to be *perpendicular* to each other and the angles thus formed are called *right angles*. If the two adjacent angles are not equal, the angle which is greater than the right angle but less than 180° is called an *obtuse* angle and the lesser one is called an *acute* angle.

We may consider the formation of an angle in another way, which perhaps gives more meaning to the term.



FIGURE 63

We may think of the straight line OM as revolving about the point O in a counter-clockwise direction. When it reaches the position ON the revolving line is said to have

passed through the *angle* MON . When it has made one-fourth of a revolution about O , the angle formed is called a *right angle*; when it has made one-half a revolution, the line ON lies in exactly the opposite direction from OM and the angle is said to be a *straight angle*; when it has made less than one-fourth of a revolution, the angle formed is called *acute*; when more than one-fourth but less than one-half, *obtuse*.

Lines which are perpendicular to the plane of the horizon are called *vertical*; those which are parallel to it are called *horizontal* lines; those which are neither parallel nor perpendicular to it are called *oblique* lines.

Two-dimensional terms. — That which has two dimensions is called a *surface*. A surface may be such that a straight line between any two points on the surface has all its points on the surface; such a surface is said to be a *plane surface* or a *plane*. If the surface is such that the

straight line connecting any two of its points does not touch the surface at every point, it is said to be a *curved surface*.

A portion of a plane surface may be inclosed by straight lines. The least number by which it can be so inclosed is three; it may also be bounded by four, five, or any higher number of straight lines or by a curved line. Such an inclosed plane is called a *plane figure* or a *polygon*.

A plane surface bounded by three straight lines is called a *triangle*. A triangle may have all three sides equal, two sides equal, or no two sides equal, giving the *equilateral* triangle, the *isosceles* triangle, and the *scalene* triangle, respectively. Triangles also differ as to angles: A triangle having one right angle is called a *right* triangle; having one obtuse angle, an *obtuse* triangle; having three acute angles, an *acute* triangle. The side upon which a triangle rests is called a *base*. The perpendicular distance from the opposite vertex to the base is called the *altitude*.

Any side of a triangle may be used as the base, and to each base an altitude can be drawn. Thus, in Figure 64, the altitudes AD , BE , and CF are drawn to bases a , b , and c , respectively.

A plane surface bounded by four straight lines is called a *quadrilateral*. A quadrilateral may have both pairs of its opposite sides parallel, one pair and but one pair parallel, or neither pair parallel, giving rise to the *parallel*

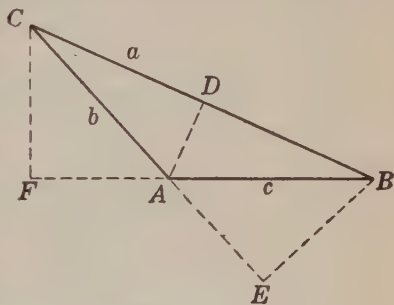


FIGURE 64

ogram, the *trapezoid*,¹ and *trapezium*,¹ respectively. A parallelogram may have all of its angles right angles, or none of its angles right angles, giving rise to the *rectangle* and *rhomboid*. A rectangle having all its sides equal is called a *square*; a rectangle whose sides are not all equal is called an *oblong*. A rhomboid having all its sides equal is called a *rhombus*; a rhomboid which does not have its sides all equal we shall call a *rhombogram*.²

Either pair of opposite sides of a parallelogram may be regarded as its *bases*; the perpendicular distance between its parallel sides is called its *altitude*. The parallel sides of a trapezoid are called its *bases*. The perpendicular distance between them is called its *altitude*. A line connecting opposite vertices in any quadrilateral is called a *diagonal*.

A plane figure bounded by five straight lines is called a *pentagon*; by six, a *hexagon*; by seven, a *heptagon*; by eight, an *octagon*; and so on.

A polygon that has its sides and angles all equal is called a *regular polygon*; not equal, an *irregular polygon*. The perpendicular distance from the center to one side of a regular polygon is called its *apothem*. The distance around any polygon is called its *perimeter*.

We may think of a regular polygon as bounded by an infinite number of sides. Such a figure is called a *circle*. A circle may also be defined as a plane figure bounded by a curved line every point of which is equally distant from a point within called the *center*.³ The length of

¹ The meanings of trapezoid and trapezium are interchanged in England.

² This name was originated several years ago by a student at the New York Training School for Teachers.

³ The term *circle* is used in referring either to the curve or to the part of the plane inclosed by it.

the bounding line of a circle is called its *circumference*. A line through the center of a circle and terminated by the circumference is called a *diameter*; a line from the center to any point on the circumference is a *radius*.

Three-dimensional terms. — That which has three dimensions is called a *solid*. The bounding surfaces of a solid are called its *faces*. The straight lines formed by the intersection of two faces are called *edges*. A point formed by the intersection of three or more edges is called a *vertex*. The face upon which a solid rests is called its *base*.

A solid which has two parallel and congruent plane figures as bases and its lateral faces parallelograms is called a *prism*. A circular prism is called a *cylinder*. A prism whose faces are perpendicular to the planes of its bases is a *right prism*. Prisms are also named according to their bases — *triangular prisms*, *square prisms*, *pentagonal prisms*, and so on. A solid which has but one base, and whose lateral faces are triangles is called a *pyramid*. If its lateral faces are all equal, it is called a *regular pyramid*; if not equal, an *irregular pyramid*. Pyramids are also named according to their bases — *triangular pyramids*, *square pyramids*, and so on. The *altitude of a prism* is the perpendicular distance between the two bases; the *altitude of a pyramid* is the perpendicular distance from the opposite vertex to the base. The straight line drawn from the vertex perpendicular to one edge of the base is called the *slant height of the pyramid*.

A solid may have all its faces, including its bases, congruent polygons, or it may not; this gives rise to the *regular solid* and the *irregular solid* respectively. A

regular solid bounded by six squares is called a *cube*. Other regular solids are not used in arithmetic, but are discussed in geometry.

The regular solid which is bounded by an infinite number of faces is called a *sphere*. A sphere may also be defined as a solid bounded by a curved surface every point of which is equally distant from a point within called the *center*. The straight line passing through the

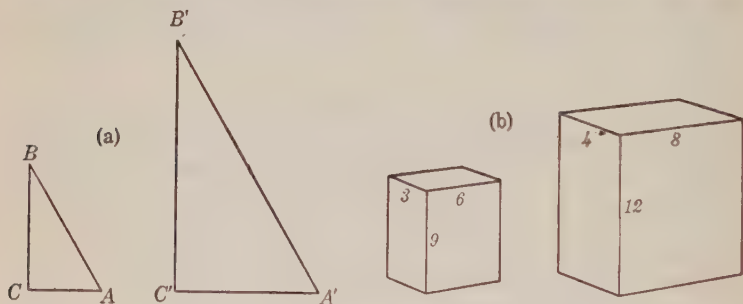


FIGURE 65

center of a sphere and terminating in its surface is called its *diameter*; from the center to the surface, its *radius*.

Similar figures. — Two geometrical figures are said to be similar when their angles are equal and their corresponding (or homologous) sides or edges are proportional.

Thus, the triangles ABC and $A'B'C'$, having angles A , B , C equal respectively to angles A' , B' , C' and $AB/A'B' = AC/A'C' = BC/B'C'$ are similar plane figures. The two rectangular solids having all their angles right angles and corresponding edges proportional ($\frac{3}{4} = \frac{6}{8} = \frac{9}{12}$) are similar solids.

DEVELOPMENT OF RULES AND PRINCIPLES

Need for rules. — Many practical problems arise which call for the measurement of lines, surfaces, and solids. In some instances it is easy to make these measurements, in others it is quite difficult. Thus, it is a comparatively simple matter to measure directly the length of a room, but not so easy to measure the surface of a sphere. A difficult measurement, however, may often be computed from one that is easy to make, by a knowledge of the relation that exists between the two. These relations are usually stated in the form of rules or principles. Thus, it would be quite difficult to measure directly the surface of a ball in square inches, but if the relation of the surface of a sphere to its circumference or diameter were known, it would be easy to measure the circumference and from this obtain the surface.

Principles or rules should, if possible, be understood before problems involving them are solved. Let us take up in order those principles which have to do with: (1) measuring lines; (2) computing areas of plane figures; (3) computing surfaces of solids; (4) computing volumes of solids. These rules are more concretely expressed by stating them in words. This should be done in the earlier grades, and when the rules are first taught. Later the rules may be expressed as formulas, but it is well for the children to translate from formula to rule and from rule to formula so that the latter may carry its full meaning.

Lines. — *Hypotenuse.* — The relation of the hypotenuse of a right triangle to the other two sides was discovered centuries ago and was first proved deductively by Pythag-

oras, for which reason it is called the Pythagorean theorem. Over seventy different deductive proofs have been worked out for this famous theorem.

We may discover inductively the relation of the hypotenuse of a right triangle to the other two sides.

Let us take the following right triangles, one having sides measuring 3 and 4 inches; the other having sides

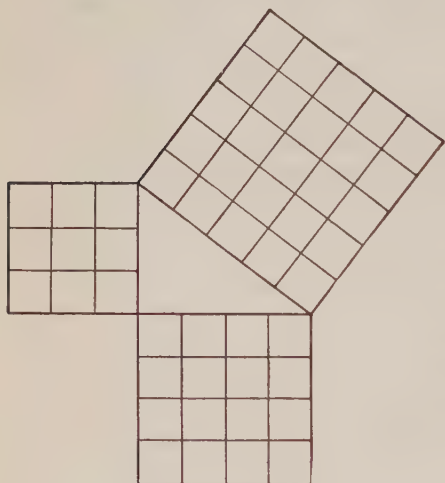


FIGURE 66

measuring 5 and 12 inches. By measurement, the hypotenuse of the first is found to be 5 inches; that of the second, 13 inches. By erecting squares on the two sides and the hypotenuse as at left, it is seen that the sum of the squares on the two sides is equal in area to the square on the hypotenuse.

Therefore it is probably true that the square of the hypotenuse is equal to the sum of the squares of the two sides.

The deductive proof is too difficult for arithmetical reasoning. (See any plane geometry.)

Formula: $H^2 = a^2 + b^2$.

Circumference. — The relation of a circumference of a circle to its diameter has been a question that has engaged the thought of mathematicians for centuries. The finding

of the circumference from the diameter by means of ruler and compasses — the so-called *squaring of the circle* — was attempted for centuries. Let us discover this relation inductively.

Take a circle having a diameter of 5 inches. On the circumference at any point, mark a point. Draw a hori-

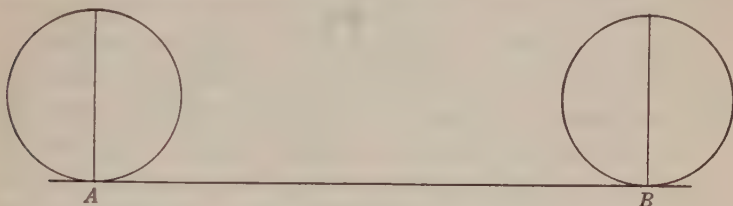


FIGURE 67

zontal line. Rest the circle on the line, marking the point where the marked point of the circle touches the line. Carefully revolve until this point of the circle again touches the line, then make a second mark on the line. The distance between these two marks on the line is the circumference. Measure this distance and divide by the length of the diameter. The result is approximately $3\frac{1}{7}$, or 3.14. The same ratio is obtained with other circles. Therefore it is probably true that the circumference of a circle is 3.14, or $3\frac{1}{7}$, times its diameter.

The deductive proof is too difficult for arithmetic. (See any plane geometry.) The value of the ratio of the circumference to the diameter is usually expressed as a decimal in ten-thousandths, 3.1416, and represented by the Greek letter π . Lambert¹ in 1761 proved that this relation is an irrational number.² The value of this

¹ BALL, W. W. — *A Short History of Mathematics*, p. 410.

² An irrational number is neither an integer nor a fraction. The following development makes clear the distinction. We may consider how many times

number has now been computed to several hundred decimal places.

Formula: $C = \pi D$, or $C = 2 \pi R$.

Plane figures. — *The area of a rectangle.* — The inductive proof is as follows: Take two rectangles one 5 in. long and 3 in. wide, and the other 6 in. by 2 in. Covering each surface with 1 in. squares we observe¹ that in the first rectangle we have 5 sq. in. in the top row, 5 sq. in. in the second row, and 5 sq. in. in the third row, making 15 sq. in. in all. In the other rectangle we have 6 sq. in. in each of the two rows, or 12 sq. in. in all. Hence: To find the area of a rectangle, multiply the number of square units in 1 strip by the number of strips. (See Fig. 69.)

Formula: $S = b \times a$.

one quantity will contain another of the same kind. The first will contain the second an exact number of times or it will not; if it contains it an exact number of times, the resulting number is called an *integer*. If it does not contain the second an exact number of times, the two quantities have a common unit of measure or they do not, giving rise to *fractions* and *irrational numbers*. Thus:

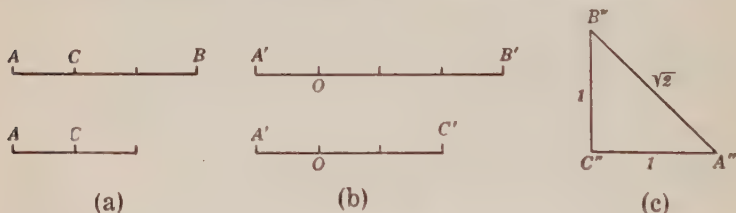


FIGURE 68

AB contains AC exactly 3 times: 3 is an integer. $A'B'$ does not contain $A'C'$ an exact number of times but the two have the common unit $A'O$, which is contained 4 times in $A'B'$ and 3 times in $A'C'$, or $A'B'$ contains $A'C'$ $\frac{3}{4}$ times. $\frac{3}{4}$ is a fraction. $A''B''$ does not contain $A''C''$ an exact number of times and the two have no common unit of measure. $A''B''$ contains $A''C''$ $\sqrt{2}$ times. $\sqrt{2}$ is an irrational number.

¹ A *square inch* may be defined as the amount of surface in a square each side of which is an inch. A *square inch* of surface may assume any form — a circle, a triangle, etc.



(a)



(b)

FIGURE 69

Area of triangle. — The inductive proof follows: Let us take the triangles ABC and DEF . Cut another triangle of size ABC and cut along AO , the perpendicular from A to BC . Fit the cut triangle into position as shown at right, forming the rectangle. Proceed in the same way with DEF .

Hence, it is probably true that the area of a triangle is one-half the area of a rectangle having the same base and altitude as the triangle.

For the deductive proof see any plane geometry.

Formula: $S = \frac{1}{2} \times b \times a$.

Area of parallelogram. — The inductive proof follows: Cut off triangle ABO on line AO and fit it into position CDO , thus forming a rectangle of same base and altitude.

Proceed in similar way with other parallelograms. Infer the following rule:

The area of a parallelogram is equal to the area of a rectangle having the same base and altitude.

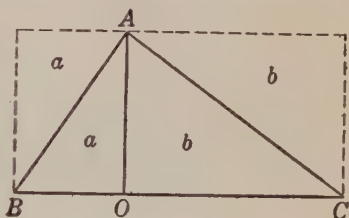


FIGURE 70

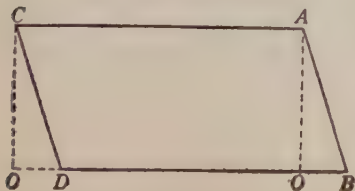


FIGURE 71

For the deductive proof, see any plane geometry.

Formula: $S = b \times a$.

Area of trapezoid. — The inductive proof follows: Cut out two trapezoids of the same size and cut one of them

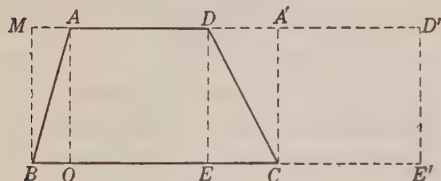


FIGURE 72

along lines AO and DE (the altitudes) into 3 parts. Fit ABO and DEC into positions MAB and $DA'C$, and $AOED$ into position $A'C'E'D'$, thus forming from the two

equal trapezoids a rectangle whose base is the sum of the two bases of the trapezoid, and whose altitude is the altitude of the trapezoid. Proceed in the same way with other trapezoids and infer the following rule:

The area of a trapezoid is one-half the area of a rectangle whose base is the sum of the bases of the trapezoid and whose altitude is the same as that of the trapezoid.

For the deductive proof see any plane geometry.

Formula: $S = \frac{1}{2} \times (b + b') \times a$.

Area of regular polygon. — The inductive proof follows: Cut out two regular polygons of same size and fit together



(a)



(b)

FIGURE 73

so as to form a rectangle, having as its base the perimeter of the polygon and as its altitude the apothem of the polygon.

Proceed in the same way with other regular polygons. Infer the rule:

The area of a regular polygon is one-half the area of a rectangle whose base is the perimeter and whose altitude is the apothem of the regular polygon.

Formula: $S = \frac{1}{2} \times p \times a$.

The area of an irregular polygon is the sum of the areas of the triangles into which it may be divided.

Area of circle. — The inductive proof follows: Cut out two circles of the same size, then cut from the center

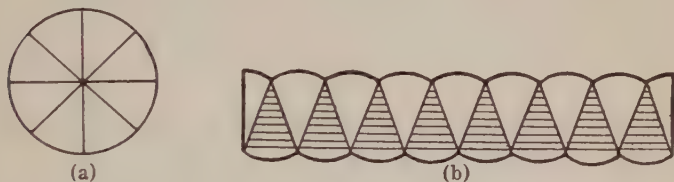


FIGURE 74

as shown in the illustration just above; the lines for cutting may easily be found by folding the circle in half, then in half again, etc.

Fit the parts of the two circles together, thus forming a figure closely approaching a rectangle whose base is approximately the circumference of the circle and whose altitude is the radius of the circle.

Proceed in a similar way with other circles. Infer the rule:

The area of a circle is one-half the area of a rectangle whose base is the circumference and whose altitude is the radius of the circle. ($S = \frac{1}{2} \times 2 \pi R \times R$.)

Formula: $S = \pi R^2$.

Surfaces of solids. — The surface of a solid, exclusive of its bases, is called its *convex* or *lateral surface*.

Convex surface of a prism and cylinder. — Let us take a square prism made of pasteboard, detach the bases and unfold as indicated in the figure below.

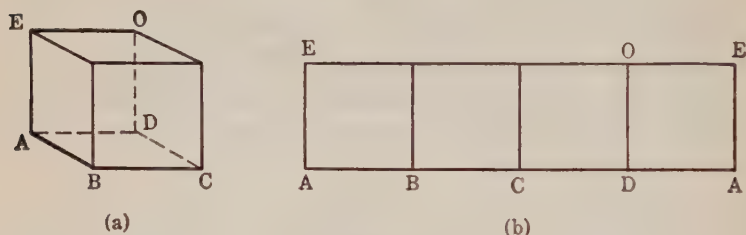


FIGURE 75

Proceed in the same way with other prisms and infer the rule :

The convex surface of a prism is the same as the area of a rectangle whose base is the perimeter of the base of the solid and whose altitude is the altitude of the solid.

We may proceed in the same way with the cylinder, the perimeter becoming the circumference. Infer the following rule :

The convex surface of a cylinder is equal to the area of a rectangle whose base is the circumference of the cylinder and whose altitude is the altitude of the cylinder.

Formulas : For the prism : $S = p \times a$.

For the cylinder : $S = C \times a$.

Lateral surface of a regular pyramid and cone. — Let us construct a pyramid, cut, and unroll as shown in Figure 76.

The lateral surface is made up of triangles, which together form half of a rectangle whose base is the perimeter

of the base of the pyramid and whose altitude is the slant height of the pyramid. Proceeding in the same way with other regular pyramids, we may infer the following rule :

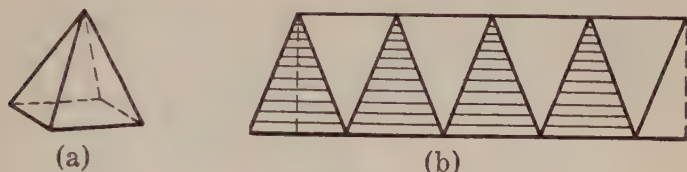


FIGURE 76

The lateral surface of a (right) pyramid is one-half the area of a rectangle whose base is the perimeter of the base of the pyramid and whose altitude is the slant height of the pyramid.

We may proceed in the same way with the cone and arrive at the following probable rule : The convex surface of a cone is equal to one-half the area of a rectangle whose base is the circumference of the base of the cone and whose altitude is the slant height of the cone.

Formulas :

For the pyramid : $S = \frac{1}{2} \times p \times s$.

For the cone : $S = \frac{1}{2} \times c \times s$.

Surface of a sphere. — The inductive proof follows : Take half a sphere, drive a nail into the center of the curved surface and another into the center of the plane surface and wind a cord around each nail until both surfaces are covered. Comparing the lengths of the cords, it will be found that the length of the cord which covered the curved surface is twice the length of that which covered the plane.

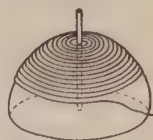
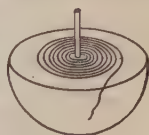


FIGURE 77

Proceed in a similar way with other hemispheres, and infer the following rule :

The surface of a sphere is four times the surface of a circle of the same radius.

Formula: $S = 4 \pi R^2$.

Volumes. — Many practical problems arise in which it is necessary to find the cubical contents of solids, such

as finding the number of cubic feet in a room, the cubical contents of bins, tanks, measures of capacity, etc.

Volume of prism and rectangular solid. — Let us take a rectangular solid 4 in. long, 2 in. wide, and 3 in. high and cover the base with inch cubes; in this layer there will be 4 cu. in. in 1 row and 2 such rows. If we pile up 3 such layers, the solid will be filled.

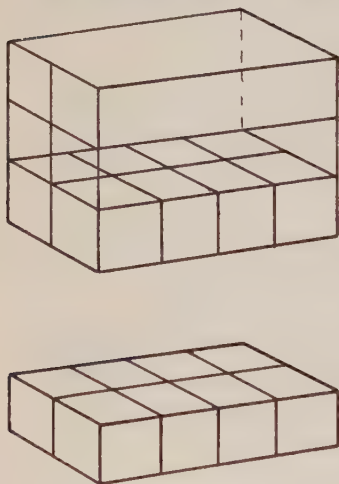


FIGURE 78

and square prisms and infer the following rule :

The volume of a rectangular solid equals the number of cubic units in 1 layer multiplied by the number of layers. The number of cubic units in 1 layer is found by multiplying the number of cubic units in one row of the layer by the number of rows.

Formulas: For prism: $V = l \times w \times a$.

For cylinder: $V = \pi \times R^2 \times a$.

Right pyramid and cone. — The inductive proof follows: Construct any regular pyramid with open base; construct a prism of same base and altitude.

Fill the pyramid with sand or rice or some dry commodity and empty into the prism; continue until the prism is full. Do the same with cone and cylinder, and proceed in the same way with other pyramids and prisms

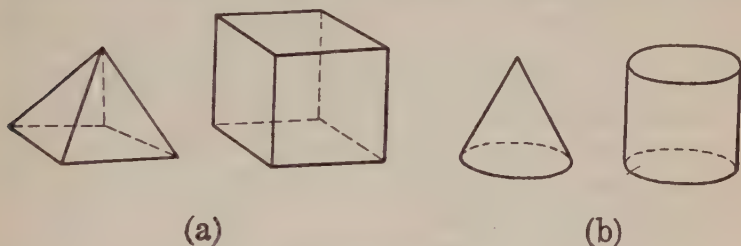


FIGURE 79

and with other cones and cylinders. It will be found that 3 fillings of the pyramid will fill the prism, and 3 fillings of the cone will fill the cylinder. Infer the rules:

The volume of a right pyramid is one-third the volume of a prism having the same base and altitude. The volume of a cone is one-third the volume of a cylinder having the same base and altitude.

Formula: $V = \frac{1}{3} \times B \times a$.

Volume of sphere. — For apparatus we may use a wooden sphere cut up into a number of small pyramids whose common vertex is the center of the sphere. Or a picture may be made of such an apparatus.

Hence the volume of the sphere is seen to be the sum of the volumes of all the pyramids whose bases will form

the surface of the sphere and whose common altitude is the radius of the sphere. As the volume of any pyramid

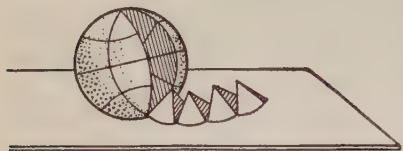


FIGURE 80

is one-third the volume of a prism of same base and altitude, we may infer the following rule :

The volume of a sphere is one-third the volume of a prism whose base is

the surface of the sphere and whose altitude is the radius.

As the surface of a sphere equals 4 times the product of the square of the radius by 3.1416, the volume may be expressed in terms of the radius, thus: $\frac{1}{3} \times 4 \times R^2 \times 3.1416 \times R$, or $\frac{4}{3} \pi R^3$.

The number of cubic units in the volume of a sphere is $\frac{4}{3}$ times the cube of the number of units in the radius times 3.1416.

Formula: $V = \frac{4}{3} \pi R^3$.

Laws of similarity. — In similar figures corresponding sides are proportional (by definition).

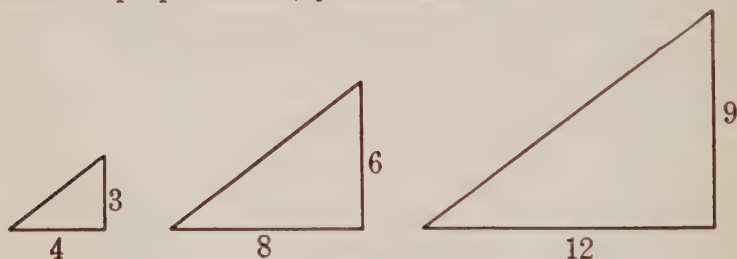


FIGURE 81

Let us discover the relation of the areas of similar figures. Consider the similar triangles in Figure 81,

having bases 4'', 8'', and 12'', respectively, and altitudes 3'', 6'', and 9'' respectively.

The area of the first triangle is 6 square inches; of the second, 24 square inches; and of the third, 54 square inches. Examining these results we discover that when the corresponding linear parts are twice as great, the area is four times as great; when three times as great, the area is nine times as great. Hence we may infer it is probably true that: *In similar figures areas are proportional to the squares of corresponding linear parts.*

Proceeding in a similar way we discover that: *In similar figures volumes are proportional to the cubes of corresponding linear parts.*

CONCRETE USES OF GEOMETRIC FIGURES AND MEASUREMENTS

Geometric figures in designs. — Most children take delight in making interesting combinations of geometric forms, in seeing what kinds of figures they can draw with ruler and compasses. Often it is the beauty of the figures that arouses interest, sometimes it is merely their curiosity. In either case the child that can be induced to play with his drawing instruments, to copy or to invent interesting and beautiful designs, is sure to acquire by this activity a mass of visual images that will stand him in good stead when he takes up the formal study of geometry; moreover, he will be more likely to observe and appreciate these forms in architecture and other forms of art. Modern textbooks, especially those in

Junior High School mathematics, give conscious attention to this aspect of inventional and intuitive geometry.¹

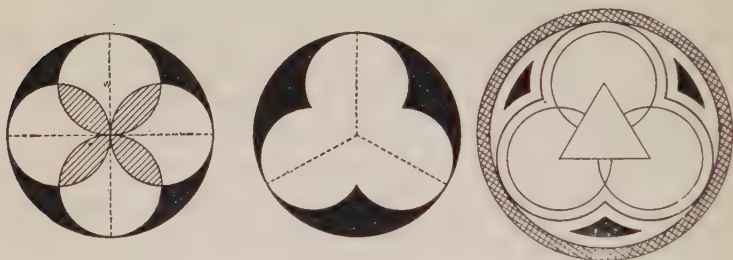


FIGURE 82

Problems in geometric measurements. — An endless variety of problems arises in various occupations, which involve practical measurements of lines, planes, and solids.

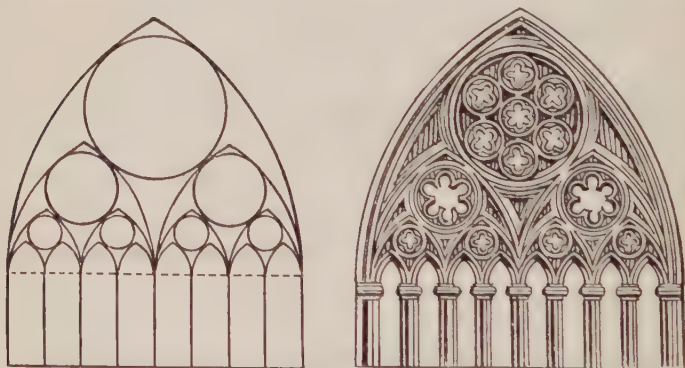


FIGURE 83

The solutions of such problems depend upon a knowledge of the rules involved. In most problems, the direct

¹ BRESLICH, E. R. — *Junior Mathematics*, I, Chapters V and VII; The Macmillan Company.

application of the rule is called for and these problems may all be solved by arithmetical procedure. If the relation involved is indirect, the problem properly belongs in those grades in which the pupils have experience in interpreting formulas and in solving algebraic equations.

In the lower grades, the concrete form of the rule as stated in the inductive proofs is better; in higher grades, the formula may be used. It is, strictly speaking, not true that the area of a rectangle equals the length times the width; for length and width must be expressed as concrete numbers; thus, "The length is 6 *ft.*; the width is 4 *ft.*" This would bring us to the multiplication of 6 *ft.* by 4 *ft.*, which, as we have learned in multiplication, is impossible (p. 77). But when a pupil has learned to use a formula he understands that in $A = l \times w$, A stands for the *number* of square units in the area, and l and w also stand for abstract numbers, the *number* of linear units in the length and width respectively.

Many interesting problems involving space relations arise which may be solved through a knowledge of the laws of similar figures. Problems in indirect measurements, such as computing the heights of buildings from the lengths of their shadows or from sighting the top of a building along the hypotenuse of a right triangle, the distance of an inaccessible object or the width of a stream, may be solved by applying the laws of similarity.

Problem (a) (p. 366) illustrates a problem of ordinary difficulty that may be solved in intermediate grades; problem (b) illustrates a somewhat difficult type in which a knowledge of solving an equation and of extracting square root is needed, and where a picture of the figure

involved is of help in disclosing the relations; (c), (d), and (e) illustrate problems involving the laws of similarity.

(a) A floor 18 ft. long and 14 ft. wide is to be covered with linoleum; how many square yards will be needed?

L, 18 ft.	18	9)252
W, 14 ft.	14	28
Area in	72	
sq. yd., ?	18	
	252	

1 str., 18 sq. ft.

Area, 252 sq. ft.

or 28 sq. yd. *Ans.*

In the intermediate grades, the child thinks of the number of square feet in each strip. The simple problems are: How many square feet in 1 strip of the rectangle 18 ft. long? *Ans.*, 18 sq. ft. How many in 14 such strips? *Ans.*, 252. How many square yards in 252 sq. ft.? *Ans.*, 28.

(b) A steeple in the form of a square pyramid, each edge of which is 18 ft., is to be painted. For how many square feet must the painter charge if the top of the steeple is 30 ft. above its base?

One edge b., 18 ft.	$h^2 = 900 + 81$	31.3
Alt., 30 ft.	$h^2 = 981$	36
Lat. sur., ?	$h = 31.3$	1878
		939
		1126.8
Perim. b., 72 ft.	$LS = \frac{1}{2} \times 72 \times 31.3$	
Slant ht., 31.3 ft.	$LS = 1126.8$	
Lat. sur., 1126.8 sq. ft. <i>Ans.</i>		

$$\begin{array}{r}
 981.00 \overline{) 31.3} \\
 \underline{9} \\
 61 \overline{) 81} \\
 \underline{61} \\
 623 \overline{) 2000}
 \end{array}$$

In (b) the terms used in finding lateral surface are not known and must be found from other rules which involve these relations. A picture discloses the relation that the slant height forms the hypotenuse of a right triangle, one side of which is the altitude of the pyramid, or 30 feet, and the other side of which is one-half the length of one side of the base, or 9 feet.

(c) When the length of the shadow of a person 5 ft. 6 in. tall is 1 ft. 6 in., what is the height of a building whose shadow at the same time measures 9 ft.?

Ht. per., 5.5 ft.	$5.5 : 1.5 = x : 9$
L. sh. per., 1.5 ft.	
L. sh. bldg., 9 ft.	
Ht. bldg., ?	$x = \frac{5.5 \times 9}{1.5} = 33$
Ht. bldg., 33 ft. Ans.	3

(d) A person 5 ft. 6 in. tall holding a small right triangular pasteboard of which the base is 3 in. and the altitude 4 in. sights the top of a building along the hypotenuse when holding the triangle on a level with his eye. If he is 24 ft. from the building, what is the height of the building?

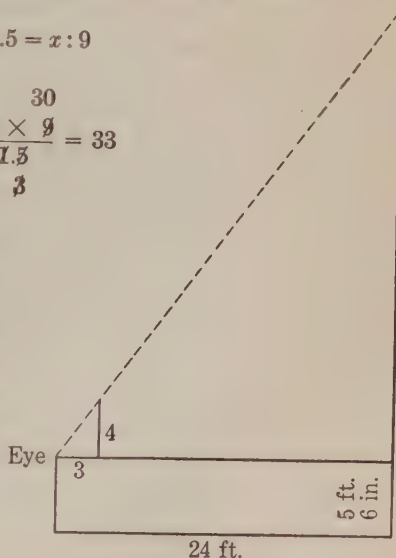


FIGURE 84

$$3 : 4 = 24 : x$$

$$x = \frac{4 \times 24}{3} = 32$$

Base tr., 3 in.
Alt. tr., 4 in.
Ht. per., 5 ft. 6 in.
Dist. fr. b., 24 ft.
Ht. bldg., ?
Ht. above eye, 32 ft.
Ht. bldg., 37 ft. 6 in. Ans.

(e) If a ball 2 in. in diameter weighs 4 oz., what is the weight of a ball of like substance 1 ft. in diameter?

Diam. 1st, 2 in.

Wt. 1st, 4 oz.

Diam. 2d, 1 ft.

Wt. 2d, ?

216

4

864

16)864

54

Ratio 2d diam. to 1st diam., 6

Ratio 2d vol. to 1st vol., 216

Wt. 2d, 864 oz. or 54 lb. *Ans.*

In similar figures
volumes are propor-
tional to the cubes of
corresponding linear
parts.

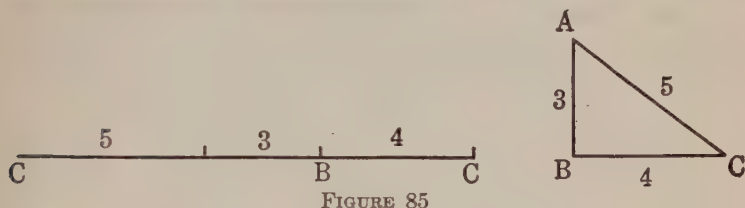
HISTORICAL NOTES

Inductive geometry. — The measurement of lines and surfaces had its origin in the measuring of land; hence the science which grew out of it came to be called *geometry* ($\gamma\eta$, land; $\mu\epsilon\tau\rho\epsilon\acute{\iota}\nu$, to measure). Its beginning was probably in Egypt, where frequent land surveys were necessary on account of the obliteration of landmarks by inundations of the Nile. The earliest Egyptian writer, Ahmes, considers finding the area of land in the form of a square, oblong, isosceles triangle, isosceles trapezoid, and circle. His formula for the area of a circle was $(d - \frac{1}{8}d)^2$; his value of π is thus seen to be 3.16^+ .

The Egyptians in building their temples were particularly interested in finding true north-and-south and east-and-west lines. By means of "rope stretchers" they first laid off a north-and-south line. This was done by using the line connecting points midway between the rising and setting points of a star. An east-and-west line was found from this by stretching a rope divided into sections 3, 4, and 5 in such a way that given the direction of AB , north and south, the direction of BC was determined as an east-

and-west line when the rope was stretched so that the end met at C . The angle at B was known to be a right angle. (Figure 85.)

It is thus evident that the property of the right triangle was known to them, as it was to the Chinese and probably the Hindus, long before the time of Pythagoras. All the



Egyptian mathematicians dealt with numerical problems and not with general theorems.

The Jews and Babylonians used 3 as the value of π . The Babylonian sign ($\star$) is supposed to be associated with the division of a circle into 6 equal parts which may have suggested the division of 360 (the Babylonian year consisted of 360 days) into 6 equal parts, thus giving rise to their sexagesimal scale. Smith¹ tells us that from the tablets found at Nippur it appears that as early as 1500 B.C. the Babylonians could find the area of a triangle, trapezoid, and possibly the area of a circle; also the volume of a parallelepiped and the volume of a cylinder.

Deductive geometry. — But none of these nations had any conception of geometry as a logical sequence of propositions. To the Greeks belongs the honor of originating deductive geometry. The first Greek to make the subject abstract was Thales of Miletus, one of the seven wise

¹ *History of Mathematics*, Vol. I, p. 40.

men of Greece, who lived about 600 B.C. He demonstrated only a few propositions, but as he was the first to give us the idea of a logical proof, he is considered one of the great founders of mathematical science.

The next great name associated with geometry as a science is Pythagoras. He is said to have been the first to prove deductively the relation of the square of the hypotenuse to the sides of a right triangle. Geometry was one of the chief subjects studied in the famous school or secret brotherhood which he established in Crotona.

A little later Plato, the famous philosopher, esteemed the subject so highly that he had inscribed over the entrance to his school, "Let none ignorant of geometry enter my door." He saw the scientific possibilities of geometry and insisted upon accuracy in statement of definitions and axioms, and upon clearness of logical proofs. He so systematized the work and laid such emphasis upon analytical proofs as greatly to influence subsequent geometers. He indignantly refused the use of mechanical appliances in geometry, thus exemplifying Smith's¹ statement: "No man ever created a mathematical theory for practical purposes alone. The applications of mathematics have generally been an afterthought."

But the greatest name in the galaxy of geometrical stars is Euclid. He arranged in book form "the mass of material which had been accumulating for two centuries following the death of Pythagoras. No doubt there were also many propositions that were original with Euclid, but the feature which made his treatise famous, and which

¹ *History of Mathematics*, Vol. I, p. 90.

accounts for the fact that it is the oldest scientific textbook still in actual use, is found in its simple but logical sequence of theorems and problems." "He was the most successful textbook writer the world has ever known, over one thousand editions of his geometry having appeared in print since 1482 and manuscripts of this work having dominated the teaching of the subject for eighteen hundred years preceding that time."¹

Euclidean geometry, as it has come to be called, has had very few changes since Euclid's day.

EXERCISES

1. Circumference of circle equals 66'', find area. Use $\pi = 3\frac{1}{7}$.
2. Surface of a sphere equals 287 sq. in. Find volume.
3. A rectangle is 3 times as long as it is wide and contains 7 acres. Find the number of rods of fence to inclose it.
4. The ratio of the length to the width of a rectangle is $\frac{3}{2}$ and the area is 294 sq. ft. Find perimeter.
5. A prism has its height 5 times its breadth and its length 3 times its breadth. Its volume is 182,505 cu. in. Find its total surface.
6. A can $4\frac{5}{8}$ '' high and $4\frac{3}{8}$ '' in diameter has a volume of how many cubic inches?
7. In making tin cans of a given volume the least tin is required when the diameter equals the height. Find the dimensions of such a can if the volume is 1 pint; if the volume is 1 gallon (8 pints). Circular ends must be cut from squares.
8. If your work for Exercise 7 is correct, the dimensions of the gallon can are twice those of the pint can. Why should this be so?
9. A schoolroom is 14' high, 27' long, and 24' wide. The hot air intake is 2' square. (a) How many cubic feet air space is given to each person when there are 35 occupants? (b) How many square feet floor space per person? (c) What must be the air velocity through the intake to furnish each person 30 cu. ft. of fresh air per minute?

¹ SMITH, D. E. — *History of Mathematics*, Vol. I, pp. 103 and 105.

10. A mold used in making little spheres of ice cream used in ice cream cones was $1\frac{1}{2}$ inches in diameter. How many cones can be filled from a gallon of cream?

11. What would be the diameter of a similar mold that would make 40 servings of ice cream from a gallon?

12. Croquet balls are made $3\frac{1}{4}$ " in diameter. Find the cost of varnish for 1 gross of balls if varnish is worth \$4.50 a gallon and a gallon will cover 400 square feet.

13. Will the weight of a box of croquet balls be greater or less than the same box filled with balls with half the diameter? Pile them as though they were cubes.

14. If a man of ideal proportions and 6 ft. tall weighs 180 pounds, how tall is a man of like proportions who weighs 130 lb.? Get result to tenths of a foot.

15. A conical heap of sand, sloping at an angle of 30 degrees from the horizontal, has a circumference of 88 yd. Find the number of cubic yards of sand. Use $\pi = \frac{22}{7}$ and $\sqrt{3} = 1.73$.

16. From the equatorial circumference of the earth compute the area of its surface in square miles.

17. Find the diameter of a 16-pound shot; of a 12-pound shot. Iron is 7.75 times as heavy as an equal volume of water, and water weighs 62.5 pounds per cubic foot.

18. Draw figures to illustrate the plane figures defined on page 348.

METHODS OF TEACHING

General suggestions. — The subject of mensuration admits of, and demands, learning by the study of concrete objects and the solution of concrete problems. It is possible for children to be practically letter perfect in the definitions and rules, and in the solution of problems like, Find the area of a triangle having a base 54" and altitude 12", and to fail utterly to meet the problem, Find the area of this (triangular) piece of tin, or this (triangular) garden patch. The most important element

in the teaching of the subject is to provide exercises requiring actual contact with the forms to be studied, and a great deal of estimating followed by careful measurement.

Lines. — The study of lines begins very early in the course, where the topic comes in incidentally with the drawing and construction work. This is the time to develop accuracy in judging the lengths of lines in feet and in inches.

Measuring the heights of the children at the beginning of the term or year, and again at the end, makes interesting work. First, have the children arrange themselves in order of height; this gives valuable practice in "more or less" judgments. Then place the six tallest with backs to the blackboard, and make marks on the board showing the heights. Have each child measure his own height and write his name and the result. Have other children repeat the measurements to check against errors. Proceed in the same way with the rest of the class. Prepare a record of the results at the beginning of the term, and then at the end each child can find out how much taller he is.

Occasionally a formal exercise in estimating and measuring is worth while. The following will suggest some of the possibilities.

(1) Each of a group of children draws a horizontal line on the board. Another group takes its place and each one tries to draw a vertical line the same length as the horizontal one. Each of a third group measures the two lines and says: "——'s vertical line is —— inches too ——."

(2) The teacher draws a line on the blackboard and says: "Let us see who can tell how long this line is. Write your answer on the slip

of paper." Have the line measured and ask each child to write on his slip. "My guess was — inches too —." Commend those whose estimates come rather close.

NOTE.— In the higher grades such an exercise could be concluded by each computing the per cent of error and by seeing how close the average estimate is to the actual measurement. In the 7th or 8th grades each child, instead of estimating, might measure with the yardstick to the nearest eighth, or sixteenth, of an inch. They will finally discover that all practical measurements are approximations and see that the degree of accuracy required will depend upon the use which is to be made of the data.

Angles. — The complete notion of what an angle is, is probably beyond the powers of children in the grades. The easiest idea is that of a right angle as a "square corner," then the acute angle as a sharper corner, and an obtuse angle as not so sharp as a right angle.

Later on it is possible to enrich this concept of angles through exercises like these:

"Luke and Henry will stand close together before the class, Luke facing west and Henry south. Now walk straight ahead. Your routes form what kind of an angle? Does the angle grow larger the farther you walk? In what other directions could you walk to form a right angle?" etc.

Exercises in estimating and measuring with the protractor the size of angles in degrees, analogous to those on page 373, are of high value; also exercises in pointing north, west, northwest, 30 degrees east of north, etc.

Rectangles. — *General notion.* — The qualities of rectangular figures are learned largely through observation and the frequent use made of rectangular objects, such as pieces of paper, book covers, post cards, floors, walls,

etc. But before a child can actually draw, or cut, a rectangle of given dimensions he must be conscious of certain characteristics, and it is more than likely that he will not be conscious of them without having his attention definitely focused upon them. These characteristics are: the corners are all square corners; opposite sides are the same length. The child's definition of a rectangle is not a logical one, for the parallelogram, of which the rectangle is a species, is as yet undefined. The same thing holds, of course, for all the geometric forms studied; when first encountered they must be defined through concrete experience. This experience, as intimated above, first takes the form of observation and use of the finished object, and later takes the form of drawing and construction exercises.

Areas. — Finding the area of rectangles is first taught along with or immediately after the table of square measures — usually in the fourth year. The essential steps are:

1. Define the meaning of *area* as an amount of surface. "On which of these pieces of paper could you work the most examples?" is a question that will fix attention upon area.

2. In measuring, we always use some *unit* of measure — for a short line, the inch; for a longer line, the foot or yard. Have we learned any unit of measure that will measure the surface of these papers? Consider the inch, quart, pound, etc. Present the *square inch* as a suitable unit.

3. Find by actual application how many square inches can be placed in a row along one side of the rectangle; then find how many such rows would cover the rectangle. Multiply to find how many square inches in all. Repeat these steps with several rectangles.

4. When the children have grasped the above steps dispense with

the physical square inches and supply the children with foot rulers. Have them *measure* to see how many square inches there would be in a row, *measure* to see how many rows there would be, etc.

5. When the children have had enough experience with such problems the rule, "Multiply the number of square units in one row by the number of rows," can give place to the formula $A = l \times w$. It is a mistake to be in a hurry about introducing the latter. The rule first stated is concrete; its use makes clear the distinction between *area* and *perimeter* (very often confused when the thought is entirely about lines); and the measurement of a rectangle takes its place as a practical, meaningful problem. Why, by the way, should a textbook, or teacher, drill children endlessly on such abstract work as finding areas of rectangles of given dimensions? For example:

LENGTH	WIDTH	AREA
5 ft.	3 ft.	?
18 in.	16 in.	?
etc.	etc.	etc.

A child will learn more from solving 20 varied, well-chosen and well-timed *real* problems than he will from solving 500 such banalities. Precede the above data with the direction, "Draw to scale and find areas . . ." and the work is redeemed.

Perimeters. — While rectangles are being studied the method of finding a perimeter may as well be learned. There are many practical applications through which to make the work real. Finding the amount of picture molding for a room, the amount of edging for a handkerchief or a bureau scarf, the amount of chicken wire for a chicken yard, the amount of border needed in papering a room, etc. — these illustrate the kind of situation from which problems can be made. Probably the easiest way to present the subject is to give the pupils

some easy concrete problems to solve. Let them work out the solution in their own way.

Suppose the first problem is: "How many feet of picture molding must we buy to go around our schoolroom?"

One child might take a tape measure and simply measure around the room and announce the result. Another might measure each side separately and add. Another might measure one side and multiply by 2, then one end and multiply by 2, and then add. Before long they will, with a little help, arrive at the best method — *Add the length and the breadth and multiply by 2.*

Apply this rule to other situations immediately at hand, then to situations described in words. In the latter case scale drawings will be serviceable in making the situations vivid.

As soon as the children begin to use formulas the rule may be stated thus: $P = 2 \times (L + W)$.

Parallelograms. — *General notion.* — In the last section the point was made that geometric forms are not, at first, defined logically, but rather through objective experience. A correct notion of a parallelogram may easily be obtained by comparing it with the rectangle. The learner's attention should be focused upon the likenesses and then upon the single respect in which they differ. They are alike in that the opposite sides are equal, the opposite sides are parallel, the opposite angles are equal; they are different in that the rectangle has all its angles right angles. If time permits, or conditions demand, it is possible to bring out the comparison very clearly by using the following device.

Take the cover of a shoe box and cut out the flat top,

leaving the rest intact. By appropriate manipulations it will be shown that many parallelograms may be formed, but only one rectangle.

Area. — We need not tarry long on this topic, for much that has preceded will apply here with equal force.

In dealing with rectangles children discover that a square inch is not necessarily a *square* at all; it may be of any shape so long as it *has the same area* as a square 1 inch by 1 inch. It is now possible to show that a parallelogram can be changed to a rectangle having the same area. For this purpose no book illustration, no fancy device, however excellent, can take the place of cutting and fitting done by the children themselves.

Really practical problems with parallelograms are not easy to find. In cities and towns the streets sometimes intersect at oblique angles, giving rise to parallelograms in problems about paving streets, laying sidewalks, measuring building lots, etc. But, after all, children become interested in parallelograms when they are presented for their own sake — they are a part of the game.

Triangles. — *General notion.* — Children have very little occasion to use triangles. Where city streets meet at sharp angles there are likely to be triangular plots of ground and *flatiron* buildings. The equilateral triangle occurs in conventional designs used in architecture and elsewhere; and it is useful when irregular polygons are cut up into triangles for the purpose of finding areas. Their limited use in ordinary affairs does not, however, rob them of interest.

Probably the study of triangles should begin with observation and construction of (1) equilateral triangles,

(2) isosceles triangles, (3) scalene triangles (or just triangles). In connection with the equilateral triangle, the children are always interested to see how six such triangles fit together to make a hexagon, and how the hexagon fits into the circle. The relationship which obtains between the regular polygon and the circle, and the dissection of such polygons into isosceles triangles is of absorbing interest to many children, and constructive work along these lines is easy for most of them and forms a good basis for later work.

Area. — There are three methods of developing the rule for finding the area of a triangle :

1. Give each child a triangle (all the same size and shape). Have them fit two triangles together so as to make a parallelogram. Infer that a triangle is one-half of a parallelogram having the same base and altitude.
2. Give the children parallelograms of different sizes and shapes. Have them draw, then cut along a diagonal. Infer and conclude as in (1).
3. Proceed according to the development explained on page 355.

After the rule, or formula, has been developed and a few problems have been worked, it is a good plan to hand out paper triangles and ask the children to make the proper measurements and compute the area, doing the work on the triangle itself. Then have them turn the triangle over and, using a different side as the base, find the area a second time. Then compare the results.

The right triangle. — The experiments with the triangles on page 352 may be supplemented with experiments with right triangles with *any* length of sides. Of course the formula $h^2 = a^2 + b^2$ will in such cases be satisfied only

approximately. The technique here suggested will not only carry conviction regarding the truth of the principle but it will afford valuable practice in the use of squared paper — a much-used mathematical instrument.

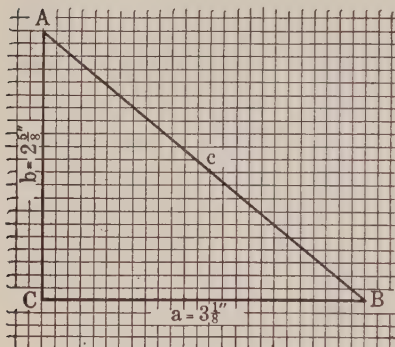


FIGURE 86

Supply the pupils with compasses and with squared paper scaled in eighths of an inch or smaller units. Each pupil draws a right triangle with the perpendicular sides of any length he pleases expressed in eighths of an inch. If he uses

the figure here shown, he will record his work thus:

$$\text{Side } a = 3\frac{1}{8}$$

$$\text{Side } b = 2\frac{5}{8}$$

$$\text{Side } c = AB = 4\frac{3}{4} \text{ (approximately), by measure.}$$

By the formula,

$$c^2 = \left(\frac{25}{8}\right)^2 + \left(\frac{21}{8}\right)^2 = \frac{1096}{64}$$

$$c = \frac{32.6}{8} = 4\frac{3}{8}$$

The results obtained by measurement and by use of the formula are seen to differ by only $\frac{3}{160}$ of an inch, — less than $\frac{1}{10}$ of an inch.

$$\begin{array}{r} 10'66 \overline{)32.6} \\ 9 \\ \hline 62 \overline{)166} \\ 124 \\ \hline 646 \overline{)4200} \\ 3876 \\ \hline 324 \end{array}$$

$$\begin{array}{r} 8 \overline{)32.6} \\ 4\frac{6}{8} = 4\frac{3}{4} \\ \hline 3\frac{3}{4} \quad 15 \\ \hline 4\frac{3}{8} \quad 12 \\ \hline 16\frac{3}{8} \quad 18\frac{3}{8} \end{array}$$

Some teachers hesitate to use a method such as the one just described because it takes so much time. They argue

that in the time required for this experiment the pupils could solve possibly five or ten exercises in application of the rule. But we are becoming more convinced all the time that it is not mere frequency but vividness and intensity that make for the mastery of new ideas, and also for their retention.

Circles. — *Relation of circumference to diameter.* — In getting the relation between the circumference and diameter the children can measure as accurately as possible the circumference and the diameter of several circular objects and divide each circumference by its diameter. They will observe that the quotients are approximately the same. If the work is carefully done, the average should be true to two places (3.14...). The work should be arranged in tabular form, thus :

OBJECT	CIRCUM.	DIAM.	$C \div D$
Trash basket	_____	_____	_____
Tin can	_____	_____	_____
Carton	_____	_____	_____
Ink bottle	_____	_____	_____
Etc.	_____	_____	_____

Another method is to have each child cut from heavy cardboard a circular disk of any size he pleases, make a pinhole in the center, and draw a diameter. Then proceed as explained on page 353. If the work is carefully done, the ratio $\frac{C}{D}$ will be found to equal about $3\frac{1}{7}$. The exact quotient cannot be expressed by any fraction, common or decimal.

Finding area. — Proceed as described on page 357, and infer that $A = \frac{1}{2} \times C \times r = \frac{1}{2} \times 2 \times \pi \times r \times r = \pi r^2$.

It is clear that this development requires a fair working knowledge of equations; hence, in most schools it is suitable only for the eighth grade.

Another method is to use squared paper. In the figure the radius is 10, hence the square on the radius contains 100 small squares, that is,

$$r^2 = 100.$$

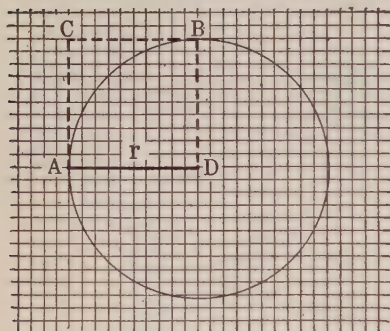


FIGURE 87

Count the squares entirely within the quadrant (quarter circle), then combine the parts of squares to get a close estimate of the area of the quadrant. Multiply this result by 4. Find how many times the area of the circle contains r^2 (100). The easiest way is to point off two places in the area of the circle which will be found to be about 314. 310 to 318 would be fairly good.

Hence we infer that the area of a circle is about 3.14 times the square on the radius, or $A = 3.14 r^2$.

NOTE. — Squared paper will be found useful in teaching other topics in this chapter.

Rectangular solids. — *Properties.* — It is customary to have the children study physical solids like inch cubes, wooden blocks, pasteboard boxes, etc., for the purpose of bringing out the following facts: There are six faces — opposite faces being equal; there are twelve edges — three sets of four equal edges; there are eight corners. It may help to take a pasteboard box and cut along the

edges so that the six faces can be laid out flat on a table ; then reverse the process, folding the faces so as to get the box. If more objective work is needed, it may take the form of constructing a box of given dimensions from a plain sheet of heavy paper. The study of the classroom involves viewing such a solid from the standpoint of an *insider*; here again the faces, edges, and corners should be identified.

Finding the superficial area. — The study described above leads easily into the method of getting the area. From the developed surface of a rectangular solid it is seen that the four sides combine to form one rectangle, and the ends are two equal rectangles. The method of finding the area is to proceed as on page 358 and to increase this result by the end faces.

Another rule grows out of the fact that there are three pairs of equal surfaces. Find the area of one surface in each pair, add these results, and multiply by 2. This is analogous to the method of finding the perimeter of a rectangle.

In practical situations it is seldom necessary to find the total area of a rectangular solid ; in papering we deal with four, or five, surfaces, in plastering with five surfaces of the room, etc. However, the formal study of the subject combined with plenty of practical problems in which judgment about the number of surfaces must be exercised is better than either of these aspects of the work taken separately.

Finding the volume. — The method described on page 360 is so simple that it is unnecessary to discuss it in detail. The teaching suggestions on page 375 have their analogies

here. It may stimulate interest to show two boxes of very nearly the same volume, but quite different in shape, and ask which of them is larger. It is well at some point in the development to bring out the general principle that the thing to be measured and the unit of measure must always be alike in quality. Thus, a line (length) must be measured with a unit having length, a surface (length and breadth) must be measured with a unit having length and breadth, a solid (length, breadth, and height) must be measured by a unit having these three dimensions.

The development starts with a whole to be measured, selects a unit of measure, and then proceeds with the discovery of the relationship. A second method of approach consists in starting with a supply of inch cubes and building up some rectangular solid. The analysis of the process brings out the relationship expressed in the rule. It is impossible to say which of these is the better for a first lesson; if the first method is used, no other is needed, but it may be that the second method of approach followed by the first method mentioned would make the stronger impression. Whichever method is used, keep in mind that what the pupil does for himself is remembered more vividly than what he merely observes.

Finding the volume of a rectangular solid is a problem of rather frequent occurrence, but rarely does it appear in its simplest form; it is much more likely to occur as a step in a larger complex problem. For example, we have the problems:

How many tons of coal will my bin hold? How shall I build my bin so as to hold 12 tons of coal? How many tons of hay in a certain mow

or stack? What is the capacity in gallons of this tank? Does our schoolroom meet the standard requirement concerning air-space per pupil? At what rate must the air flow through the intake to supply each occupant with the amount of air required by accepted standards? How many cubic yards of concrete are required for a paving or construction job? The teacher who is alert will find many interesting applications and will encourage the children in discovering others, both in their environment and in their reading.

Cylinders. — The logical development of the facts and principles has been given on pages 349 and 358. The procedure in teaching these facts and principles is so similar to that described under the teaching of rectangular solids that no further discussion is necessary.

Problems involving cylinders have to do with tin cans, paper cartons, pails, silos, tanks, pipes, wire, wire nails, etc. Problems about surface area occur in estimating the cost of painting a silo, tank, or pipe, in figuring the amount of lumber for building a silo or cistern, or the amount of sheet iron for some piping, or the amount of paper for the labels used on the tin cans consumed in a cannery, etc. Problems about volume occur in figuring the rate of flow through water pipes, the capacity of a silo or other cylindrical container, the amount of water in a well, etc. Cylinders occur in every child's environment, and there is no excuse for neglecting the practical side of the topic.

Irregular forms and special methods. — Some practice in the analysis of irregular plane figures into the type forms treated in this chapter is desirable if time permits. The end elevation of a barn is the combination of a rectangle and a triangle; the lumber for the siding or the quantity of paint for covering this area suggest themselves as appropriate problems.

Decorative designs are often combinations of type forms. A square-headed bolt is a combination of cylinder and prism. A square tower is capped by a pyramid and a round tower by a cone. An iron nut is a prism with a cylinder cut out of it.

In teaching these combinations the chief interest and value lie in the discovery of the forms in the pupils' environment and in the analysis of the forms into their geometric elements. Of course a few of the problems should be worked out in detail.

Where the solid is not a combination of type forms, the method of volume displacement may be used. Submerge the object in a vessel of water and measure the volume of water displaced — which is a type solid.

EXERCISES

1. Prepare and conduct a ten-minute exercise on each of the following : (a) estimating the lengths of lines ; (b) estimating the size of angles.

2. Prepare a list of exercises and problems about the perimeter and area of rectangles. Explain how you would conduct class work on them.

3. Prepare in detail a first lesson on finding the area of a triangle.

4. Try the two methods (p. 381) of finding the ratio of circumference to diameter. Discuss the advantages and disadvantages of each as a teaching method.

5. Discuss the relative merits of the two methods of developing the rule for finding the area of a circle (pp. 357 and 382).

6. Outline a first lesson on rectangular solids suitable to the 6th grade.

7. Plan a first lesson on finding the volume of a rectangular solid using each method described on pages 384, 385.

8. List in teaching order the topics to be taught under cylinders and state briefly how you would present each topic or lesson.

9. Let each member of the class find a few practical problems in mensuration or some interesting examples of geometric forms occurring in his environment.

CHAPTER XIV

PERCENTAGE

TEACHER'S KNOWLEDGE

HISTORICAL NOTES

Early uses of per cent. — We have seen how slow was the development of decimal fractions (p. 217) and how mathematicians experimented with many forms of notation and methods of calculation before arriving at our present apparently permanent theory and practice of decimals. In somewhat similar fashion has been developed our concept of and practice with per cents. A few concrete facts and phases of this development are worth noting for the light they shed on present practices in teaching the subject. We quote from "Development of Arithmetic as a School Subject."¹

Problems of loss and gain, discount, interest, etc., which we now class as applications of percentage, are treated in the texts of this period (Colonial period and up to 1821), but the topic of percentage itself does not appear. Cocker (1667) first mentions per cent under profit and loss. After stating and solving three problems having to do with the absolute loss or gain he gives the problem: A draper bought 86 kerseys for 129 pounds. I demand how he must sell them per piece to gain 15 pounds in laying out 100 pounds at that rate.

The answer is justified by saying:

¹ MONROE, WALTER SCOTT — "Development of Arithmetic as a School Subject"; *United States Government Report*, Department of the Interior (Bureau of Education Bulletin, 1917, No. 10).

For as £100 is to £115 so is £129 to £148 7s; so that by the proportion above, I have found how much he must receive for the 86 kerseys to gain after the rate of £15 per C.

Nicholas Pike (1788) introduces profit and loss by saying that it "is an excellent rule by which merchants and traders discover their profit and loss *per cent* or by the gross. . . . It also instructs them to raise and fall the price of their goods so as to gain or lose so much *per cent*. . . ."

Dilworth (1743) defines "the rate *per cent*" as "a certain Sum agreed upon between the Lender and the Borrower, to be paid for every 100 Pound, for the use of the Principal, . . ." The rule is, "Multiply the principal by the rate *per cent* and divide the product by 100, the quotient is the interest required."

. . . In the general organization of the texts after Dilworth decimal fractions were placed early enough so that they might have been used directly in the solution of problems which involved "*per cent*," but in general they were not. The method of solution was accomplished by an application of the rule of three (proportion), or directions were given to divide the product by 100 or to cut off two places.

The above quotations represent the practice prior to 1821. The changes since 1821 are fairly well represented in the arithmetical texts bearing the name of Joseph Ray (1807-1855). Ray himself wrote only two or three books, but with such success that revisions and adaptations of his work were made down to as late as 1903. Relative to certain of these texts, Monroe¹ says:

In the first edition the topics that we now include under the head of percentage and its applications are given in order, simple interest, banking, discount, percentage (including profit and loss), commission, insurance, buying and selling stocks, exchange, duties, and taxes.

Rate *per cent* means rate *per hundred*; it is the sum paid for the use of 100 dollars for one year. Later *per centage* (written as two

¹ *Op. cit.*, "Development of Arithmetic as a School Subject," p. 32.

words) is defined as the group of those calculations in which reference is made to a hundred.

. . . but, in 1857 percentage (written as one word) was first presented abstractly. . . . In presenting percentage, the equivalent of common fractions in terms of per cents is given first. In solving the problems the rate per cent is to be expressed decimally. The symbol (%) is used almost exclusively instead of the words per cent. The idea of "per cent" meaning at the rate of so much on the hundred is not suggested.

DEVELOPMENT OF SUBJECT MATTER

Definitions. — From the modern point of view a *per cent* is a decimal of two places. Its logical relation to other fractions is indicated in the diagram; the fraction is classified first into common and decimal, and decimal fractions are seen to have the species *per cent*.

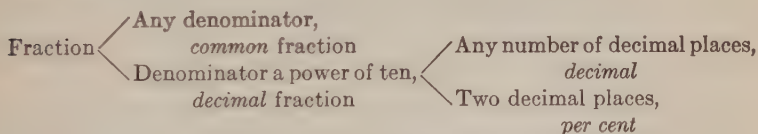


FIGURE 88

Other technical terms used in percentage are defined as follows:

The *base* is the quantity of which a per cent is taken.

The *rate*, or rate per cent, is the per cent.

The *percentage* is the result obtained by taking a per cent of the base.

The *amount* is the result of adding the percentage to the base.

The *difference* is the result of subtracting the percentage from the base.

Types of problems. — Theoretically there are thirty different types of problems in percentage.¹ It is customary, however, to teach at most only five of these cases; if these are understood, the remaining types of problems will present no difficulty.

The five types,² or cases, in percentage are:

1. To find a per cent of a number.

Or — To find the percentage from the base and rate.

2. To find what per cent one number is of another.

Or — To find the rate from the base and percentage.

3. To find a number when a per cent of it is given.

Or — To find the base from the percentage and rate.

4. To find a number which, when increased by a per cent of itself, yields a given number.

Or — To find the base from the amount and rate.

5. To find a number which, when decreased by a per cent of itself, yields a given number.

Or — To find the base from the difference and rate.

The ability to solve these five types or cases equips one to handle any situation involving per cent relationships, providing the meaning of the situation and of the technical terms used are understood. That is, if one knows percentage in general, all he needs to learn in connection with profit and loss, or commission, or commercial

¹ The truth of this statement is seen from the following three equations, from any two given terms of which the remaining three terms may be found:

$$P = B \times R \quad A = B + BR \quad D = B - BR$$

Thus, given P and R we may find (1) B , (2) A , or (3) D . With the five terms there are possible ten different combinations of two given terms: B and R , B and P , B and A , B and D , R and P , R and A , R and D , P and A , P and D , A and D . From each of these ten combinations of two given terms any one of the three remaining terms may be found. Hence we have ten times three, or thirty possible types.

² The five types in percentage correspond to the five fractional types or cases. (See page 177.)

discount, is the nature of the new situation and the practices which prevail therein.

Business fractions and their per cent equivalents. — It is often convenient to change a per cent to a common fraction and to operate with the fraction instead of a decimal. Thus, $37\frac{1}{2}\% = \frac{3}{8}$; it is easier to take $\frac{3}{8}$ of a number than to multiply it by .375. Time and energy will be saved in all cases where the equivalent fraction has small terms. The following list should be memorized :

50%, $\frac{1}{2}$	20%, $\frac{1}{5}$	$33\frac{1}{3}\%$, $\frac{1}{3}$
25%, $\frac{1}{4}$	40%, $\frac{2}{5}$	$66\frac{2}{3}\%$, $\frac{2}{3}$
75%, $\frac{3}{4}$	60%, $\frac{3}{5}$	$16\frac{2}{3}\%$, $\frac{1}{6}$
$12\frac{1}{2}\%$, $\frac{1}{8}$	80%, $\frac{4}{5}$	$83\frac{1}{3}\%$, $\frac{5}{6}$
$37\frac{1}{2}\%$, $\frac{3}{8}$	10%, $\frac{1}{10}$	$8\frac{1}{3}\%$, $\frac{1}{12}$
$62\frac{1}{2}\%$, $\frac{5}{8}$	30%, $\frac{3}{10}$	$6\frac{2}{3}\%$, $\frac{1}{15}$
$87\frac{1}{2}\%$, $\frac{7}{8}$	70%, $\frac{7}{10}$	$6\frac{1}{4}\%$, $\frac{1}{16}$
	90%, $\frac{9}{10}$	

The 7ths, 9ths, and 11ths are not often used, but are sometimes added to this list.

Methods of solving the five types. — Since the five cases in percentage correspond to the five fractional cases, the same general methods of solution can be used.

Type 1. — Mental. (a) Find $37\frac{1}{2}\%$ of 64. The steps are: $37\frac{1}{2}\%$ equals $\frac{3}{8}$; $\frac{1}{8}$ of 64 is 8; $\frac{3}{8}$ of 64 is 24. The working explanation is: $\frac{3}{8}$, 8, 24.

(b) Find 3% of 120. In mental work it is easier to first find 1% of the number and then multiply, thus: 1% of 120 is 1.2; 3% of 120 is 3.6. Working explanation: 1.2, 3.6.

(c) Find $\frac{3}{4}\%$ of 360. 1% of 360 is 3.6; $\frac{1}{4}\%$ of 360 is .9; $\frac{3}{4}\%$ of 360 is 2.7. Working explanation: 3.6, .9, 2.7.

Written. The following illustrate various forms which are acceptable in working out the examples: (a) Find 75% of 962; (b) find $\frac{3}{4}\%$ of 962; (c) find 31% of 962.

(a)

$$\begin{array}{r} 962 \\ 3 \\ \hline 4 \overline{)2886} \\ 721.5 \end{array} \quad \text{or} \quad \begin{array}{r} 3 \\ \frac{3}{4} \text{ of } 962 \\ \hline 2 \end{array} = \frac{1443}{2} = 721.5$$

(b)

$$\begin{array}{r} 9.62_x \\ 3 \\ \hline 4 \overline{)28.86} \\ 7.215 \end{array} \quad \text{or} \quad \begin{array}{r} 3 \\ \frac{3}{400} \text{ of } 9.62 \\ \hline 2 \end{array} = \frac{14.43}{2} = 7.215$$

(c)

$$\begin{array}{r} 9.62_x \\ 31 \\ \hline 962 \\ 2886 \\ \hline 298.22 \end{array} \quad \text{or} \quad \begin{array}{r} 962 \\ .31 \\ \hline 962 \\ 2886 \\ \hline 298.22 \end{array}$$

When the per cent is a business fraction, the steps are the same as in the solution of a mental problem, except it is usually better to multiply by the numerator before dividing by the denominator. In the first solution for (b) and (c) the thought of first finding 1% of the number is used; many teachers favor this method for the reason that if the children have already used this thought in the mental work, it makes for economy in learning to continue the same method in the written work.

Type 2. — The steps are: first, find what fractional part one number is of the other; then, change the fraction

to a per cent. If the numbers are small and the resulting fraction is reducible to a business fraction, the example will be worked mentally. If either of these conditions does not obtain, the corresponding step must be put into written form.

Illustrations: What per cent (a) is 18 of 30? (b) is 78 of 104? (c) of 43 is 79?

(a) The work is mental. $\frac{18}{30}$ equals $\frac{3}{5}$ equals 60%.

(b) The final step is mental if the pupil first reduces the fraction to lowest terms.

(b)		(c)
	.75	1.83
	104 $\overline{)78.00}$	43 $\overline{)79.00}$
$\frac{78}{104} = \frac{3}{4} = 75\%$ or	728	43
	<u>520</u>	<u>360</u>
	520	344
	Ans., 75%	<u>160</u>
		129
		31 Ans., 184%
		(nearly)

(c) The work is written.

Type 3. — There are two ways of analyzing this type when the problem requires written solution. To illustrate: 127.4 is 35% of what number?

	364 Ans.
3.64	35. $\overline{)127.40}$
35 $\overline{)127.4}$	105
105	<u>224</u>
224	210
210	<u>140</u>
<u>140</u>	140
140	<u>140</u>

The first solution is explained thus: If 127.4 is 35 per cent of a number, what is 1 per cent of the number?

3.64. If 3.64 is 1 per cent of a number, what is the number? 364.

The second solution uses the principle that the product of two numbers divided by one of the numbers gives the other number. $127.4 = .35 \times (?)$. To find the missing factor we divide 127.4 by the given factor, .35. This method of solution has the advantage that it is, so far as the thought is concerned, practically identical with the algebraic solution. If the teacher knows that the children are likely to take up the study of the algebraic equation in the next grade, he should by all means teach this method.

If the problem can be handled without resort to written computation, it is usually best to follow the thought of the first written solution. If the per cent is the equivalent of a business fraction, proceed as in case 3 in fractions (p. 178). To illustrate: (a) 180 is 6% of what number? If 6% of the number is 180, 1% of it is 30. If 1% of a number is 30, the number is 3000. Working explanation: 30, 3000. (b) 180 is $\frac{3}{4}$ % of what number? If $\frac{3}{4}$ % of a number is 180, $\frac{1}{4}$ % of it is 60; if $\frac{1}{4}$ % of a number is 60, 1% of it is 240; if 1% of a number is 240, the number is 24,000. Working explanation: 60, 240, 24,000.

Types 4 and 5. — A problem of type 4, or type 5, becomes one of type 3 upon increasing, or decreasing, the whole by the given per cent. To illustrate: 318 is 6 per cent more than what number? becomes 318 is 106 per cent of what number? 150 is 25% less than what number? becomes 150 is 75% of what number? The steps from this point on are the same as for type 3. In the foregoing discussion of types and their solutions we

have been concerned with the thought steps and computations necessary when the ordinary arithmetical method of solution is employed. It will be well here to consider briefly the three important methods of solution used in our schools and to illustrate each method in connection with a *practical* problem. The methods are (1) the ordinary arithmetical method of analysis, (2) the formula method, and (3) the algebraic method. For this purpose we shall take a problem of type 4.

Mrs. Howard bought a set of porch furniture at a 40 per cent reduction sale held late in the summer. The furniture cost her \$165. She was interested to tell her family how much she had saved. Compute the original price of the furniture.

BY ARITHMETICAL ANALYSIS

Cost, \$165	
Red., 40% Or. Pr.	
Orig. Pr., ?	3)165
	55
Cost, 60% Orig. Pr.	5
$\frac{1}{3}$ Orig. Pr., \$55	275
Orig. Pr., \$275	Ans.

BY FORMULA

D , \$165	
R , 40%	$\times 6.)165 \times 0.$
B , ?	275

$$B = \frac{D}{1 - R}$$

$$B = \frac{\$165}{1 - .40} = \frac{\$165}{.60}$$

$$B = \$275 \text{ Ans.}$$

BY EQUATION

C , \$165
Red., 40% Orig. Pr.
Orig. Pr., ?

$$\text{Let } P = \text{Original price}$$

$$\text{Then } P - \frac{4}{10}P = 165$$

$$\frac{6}{10}P = 165$$

$$3P = 825$$

$$P = 275$$

$$\text{Orig. Pr., } \$275 \text{ Ans.}$$

In the analytic method of solution no technicalities occur. The method consists in discovering the component simple problems and in solving each as in ordinary problems involving integers and fractions.

In the formula method one must first be sure which of the five terms, B , P , R , A , or D , are represented by the given and required terms. This preliminary step is sometimes as difficult as the solution. Having decided upon the formula needed, one must either have it in mind or derive it. To hold many formulas in mind is burdensome; to derive each formula as needed is often a more difficult task than the entire analytical solution.

The algebraic method of solution is a neat and pleasing method and may be profitably used after a few introductory lessons in solving simple algebraic equations.

Graphic aids. — Whatever method of solution is employed in the indirect cases, the learner will often need the help that comes from the objective representation of the relationships in a problem. A few illustrations are here shown.

Arthur received a 40 per cent commission for selling subscriptions to a magazine. He said to a friend, "I make \$1.10 on every subscription I sell." Find what the magazine cost the subscriber.

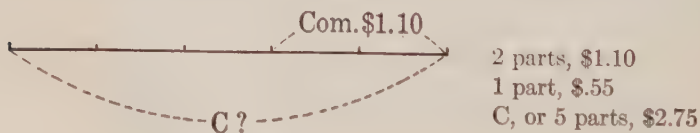


FIGURE 89

Each week Arthur sent the magazine publishers a report of his sales and inclosed the money due the publishers after his commission had

been deducted. One week he sent them \$37.95. What were his sales for that week?

3 parts, \$37.95
1 part, \$12.65
5 parts, \$63.25 *Ans.*

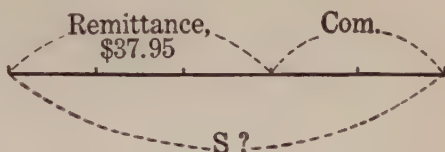


FIGURE 90

The landlord raised Mr. Enos' rent 15 per cent. If he now pays \$74.75 a month, what did he formerly pay?

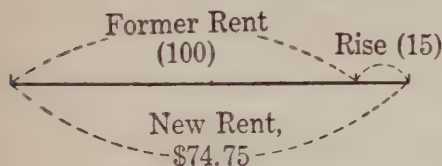


FIGURE 91

115 parts, \$74.75
1 part, \$.65
100 parts, \$65.00 *Ans.*

In the first two problems the per cent reduces to an easy business fraction; hence it is easy to represent the relations exactly. But in the third problem it is easier, and quite as serviceable, not to divide the base line into 20ths and extend it a distance equal to three of these units for the increase, but to represent the quantities approximately and to state in parentheses the number of imaginary parts into which the whole has been divided.

EXERCISES

1. Express as per cents, using the sign % :

.28	.37	.00½	.14½	.66½	.0075
.5	.2	2	3	1.5	4

2. Express as decimal fractions :

20%	45%	½%	¾%	125%
250%	375%	1⅓%	4¼%	60%

3. Express as common fractions :

$$\begin{array}{cccccc} 2\% & 2\frac{1}{2}\% & 3\frac{1}{3}\% & 133\frac{1}{3}\% & \frac{1}{2}\% \\ 175\% & 550\% & 62\frac{1}{2}\% & 87\frac{1}{2}\% & \frac{5}{8}\% \end{array}$$

4. Express as per cents, using the sign % :

$$\begin{array}{ccccccccc} \frac{1}{8} & 2 & 1\frac{1}{4} & \frac{6}{8} & 1\frac{7}{8} & \frac{5}{8} & 2\frac{3}{4} & \frac{1}{3} \\ \frac{1}{200} & \frac{3}{800} & \frac{3}{4} & \frac{1}{3} & \frac{5}{8} & \frac{1}{16} & \frac{4}{5} & \frac{5}{11} \end{array}$$

5. Call results at sight :

Of 32 : $37\frac{1}{2}\%$, 75% , $\frac{3}{4}\%$, $.75\%$, $.5\%$

Of 3500 : 20% , $\frac{1}{5}\%$, $\frac{1}{4}\%$, $.2\%$, 200% , 1000%

6. 48% of 25 = 25% of 48 = 12.

Call results at sight : 16% of 25, 16% of 2500; 72% of $66\frac{2}{3}$, 660% of 50.

7. What per cent of : 4.8 is 1.2, 34 is 68, 68 is 34, 1 yd. is 1 ft., 1 ft. is 1 yd., 1 yr. is 2 mo., 9 mo. is 1 yr.?

8. Tell of what number : (a) 65 is $8\frac{1}{3}\%$; (b) 27 is $37\frac{1}{2}\%$; (c) 42 is $66\frac{2}{3}\%$; (d) 240 is 75% ; (e) 54 is 200% ; (f) 60 is 150% ; (g) 30 is 120% .

9. What number increased by (a) 25% of itself is 55? (b) $16\frac{2}{3}\%$ of itself is 35? (c) 100% of itself is 5? (d) 300% of itself is 300? (e) 175% of itself is 66?

10. What number decreased by (a) 10% of itself is 27? (b) $37\frac{1}{2}\%$ of itself is 45? (c) $83\frac{1}{3}\%$ of itself is 7? (d) 25% of itself is 120? (e) $\frac{1}{2}\%$ of itself is 995?

11. Draw a line diagram illustrating the first and last questions in Exercises 7 to 10.

12. Give the *working explanation* for finding the following results :

(a) 5% of 90 = ?

(d) 21 is what % of 24?

(b) $83\frac{1}{3}\%$ of 54 = ?

(c) 16 is 8% of what number?

(e) $\frac{3}{4}\%$ of 240 = ?

(f) 25 is $62\frac{1}{2}\%$ of what number?

(g) 18 is $\frac{3}{4}\%$ of what number?

(h) 36 is $12\frac{1}{2}\%$ more than what number?

(i) What number increased by 8% of itself = 324?

(j) What number decreased by $16\frac{2}{3}\%$ of itself = 15?

13. Solve graphically: A is 25% taller than B. B is what per cent shorter than A?

14. The annual cost of the public schools of the U. S. is (1924) about \$1,500,000,000, a very large sum of money; but this is only a small per cent of our national income of about \$65,000,000,000. What per cent is it?

15. From 1914 to 1924 wages increased 119% while the cost of living increased only 68%. (Babson statistics.) What has been the per cent increase in the purchasing power of a day's labor? Or, what per cent more goods can a laborer buy under 1924 conditions than under 1914 conditions?

16. Get information about price indices. Interpret the following table, quoted from a magazine article:

	AVERAGE ANNUAL WAGE	BABSON'S WHOLESALE PRICE INDEX	AVERAGE RATE PER MILE	
			1 Passenger	1 Ton Freight
1911	\$ 730	88	1.96¢	.743¢
1917	1003	214	2.04¢	.707¢
1920	1820	314	2.75¢	1.052¢
1921	1665	169	3.08¢	1.275¢

In the period shown, railroad wages rose ? per cent, wholesale commodity prices ? per cent, passenger rates ? per cent, and freight rates ? per cent.

What conclusions can you draw from the above concerning the railroads?

17. The following table ¹ shows the comparative number of families receiving aid from charitable organizations in seventeen large cities in 1917 and 1921.

Study the data for St. Louis and Chicago, and supply the missing data for the other cities:

¹ Taken from a survey conducted by the *American Association for Organizing Family Social Work*, of New York, and by the *Boston Family Welfare Society*.

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	YEAR	FAMILIES UNDER CARE	FAMILIES IN WHICH DRINK WAS A FACTOR	PERCENT- AGE	PERCENT- AGE DECREASE
St. Louis Provident Association . . .	1917	3,563	412	11.6	
	1921	3,283	23	0.7	94.0
Chicago United Charities . . .	1917	7,507	625	8.3	
	1921	5,547	61	1.1	86.7
Boston Family Welfare Society . .	1917	3,589	984		
	1921	3,057	73		
Plainville, N. J., Charity Organiza- tions	1917	416	72		
	1921	525	16		
Atlantic City Welfare Bureau . .	1917	961	67		
	1921	974	12		
Newport, R. I., C. O. S.	1917	484	48		
	1921	373	12		
Portland, Maine, Assoc. Charities .	1917	277		15.5	
	1921	387		0.8	95.3
Newburgh, N. Y., Assoc. Charities .	1917	343		64.1	
	1921	432		0.5	99.1
Cleveland, Ohio, Assoc. Charities .	1917	4,571		17.1	
	1921	9,359		2.6	84.8
LaCrosse, Wis., Social Serv. Society	1917	180		25.6	
	1921	203		2.0	91.3
Portland, Ore., Pub. Welfare Bureau .	1917	1,280		0.4	
	1921	2,577		0.6	40.0 ²
N.Y. Charity Organization Soc. ¹ . .	1917		972	23.1	
	1921		196	8.4	
Hartford, Conn., Char. Organization	1917		143	27.6	
	1921		9	1.7	
Washington, D. C., Assoc. Charities .	1917		434	18.01	
	1921		67	4.5	
Rochester, N. Y., So. Welfare League .	1917		140	20.3	
	1921		34	3.8	
Providence, R. I. Soc. for Organiz- ing Charity ³ . .	1917		106	6.5	
	1921		4	0.3	

¹ Closed Cases.

² Increase.

³ Main Factor.

METHODS OF TEACHING

Terms and their meanings. — The children know something about *per cents* from their use in expressing school marks. It is necessary, however, to further define *per cent* through use in a great variety of situations and by definite association with decimals and common fractions. *Per cent* means literally *by, or on, the hundred*; but the literal translation should never be used in explaining the term to children. *Hundredths* is the better translation, for in reading 5% one does not say “5 by the hundred,” or “5 on the hundred” — he says “5 hundredths.” Much is gained by emphasizing the fact that there is very little in percentage that is new; we are dealing with the same old ideas under new names.

In the early work in changing from fractions to decimals to per cents, and inversely, the easiest exercises are those involving two-place decimals; next come those involving one-place decimals, and finally decimals of three or more orders. The steps in teaching the business fractions and their per cent equivalents are the same as in teaching the fundamental combinations: (1) get the results, (2) memorize them. In getting the results, the fractions in the table of equivalents (p. 391) may be reduced to two-place decimals and then to the common per cent form, using the sign %; or, the unit fractions *only* may be so reduced and their multiples found by multiplying, adding, or counting. For example, the children find, by the common method of reduction, that $\frac{1}{5} = 20\%$; then they can count, first by *fifths* ($\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{5}$), then by 20 per cent (20%, 40%, 60%, 80%, 100%), and finally by

the two forms alternately ($\frac{1}{5}$, 20%; $\frac{2}{5}$, 40%; etc.). A similar exercise on the *eighths* would include the *halves* and *fourths*: $\frac{1}{8}$, $12\frac{1}{2}\%$; $\frac{2}{8}$ or $\frac{1}{4}$, 25%; $\frac{3}{8}$, $37\frac{1}{2}\%$; $\frac{4}{8}$ or $\frac{1}{2}$, 50%; etc. Similarly, the *sixths* include the *thirds*, and the *tenths* include the *fifths*.

These counting exercises, relieved and enriched by concrete exercises, make excellent drill work and should be used frequently. The equivalents should be taken up in the order shown in the table so that the possible associations between the various series may be brought out. There is no reason why the counting exercises should not be extended beyond 100 per cent; familiarity with these higher per cents is often serviceable.

Diagrams like the following, used in connection with the early counting work, will assist in the formation of the desired associations, and the imaging of the space relations is quite certain to help in the solution of problems.



FIGURE 92

Problems. — The logical development of the types of problems and the general theory of their solution have already been discussed (pp. 391–397).

Of these five types the first is of more frequent occurrence in practical affairs than all the other types put together. Next in order of practical importance is type 2,

then type 3, and finally types 4 and 5, which are of about the same importance. It is not to be inferred, however, that frequency of occurrence is an index of the relative amounts of time that should be devoted to these types.

The three methods of solution. — The comparative advantages of the analytical, formula, and equation methods of solving problems in percentage have already been set forth (p. 396).

In using the formula method the child has to interpret the problem to see which of the five terms are given and what is required, then he must recall from memory the appropriate formula, and finally he must make the appropriate substitutions and simplify the expression. If the formula method is used throughout the work in percentage, and it usually is used throughout if at all, the children are under the necessity of memorizing a formula for each type of problem studied. If a formula is forgotten, or if a problem occurs which departs ever so slightly from the types that have been studied, the pupil is likely to be rendered helpless. True, analysis occurs when the formula method is used, for without analysis it would be impossible to identify the problem as coming under a given type; but there is this important difference between analysis in this case and analysis as used in the arithmetical solution — in the latter the analysis occupies the focus of attention throughout the solution, while in the former the analysis seems to be a mere preliminary to the solution.

If any indirect type is to be studied at all, it must receive enough time and attention to insure its mastery; hence it follows that the apportionment of time must be

determined by the relative difficulty of the types to be taught. The common practice of teaching rather thoroughly types 1, 2, and 3 in the first year in which percentage is studied (usually the 6th year) and then covering the whole subject, including types 4 and 5, in connection with the study of equations, probably meets most of the requirements of sound pedagogy.

In type 2, children are likely to be puzzled as to what number should be used as the divisor (base). It is unwise to habituate them to just one form of the question, for in practice it may appear in either form. Probably the best way to meet this difficulty is to teach them that the divisor, or base, always is the object of the preposition *of*.

Graphic aids are most helpful in problems of types 3, 4, and 5. This picturing of the relations involved serves to bring the situation more vividly before the child than can any purely abstract method of attack.

The advantage of the algebraic method of solution is that the indirect cases are all taken care of without making a special study of each. The first equation in the solution states a direct relation, and the answer is found by applying a few simple axioms. It certainly takes less effort to pass from a knowledge of the direct cases (finding the percentage, or amount, or difference, from the base and rate) to a mastery of the indirect cases by the study of the equation than by the study of formulas; moreover, a knowledge of these formulas is of no value in situations other than those involving per cents, while the principles of the equation find application in all sorts of situations.

The present tendency is to use analysis in the early study of percentage, and later to introduce the equation.

The content and organization of problem material. — Percentage has a wide range of general applications, and these general applications are of greater importance to people in general than are the technical applications such as commission, commercial discount, etc. For children in the grades the intricacies and usages of business practice have slight interest and meaning — only the simplest elements need be taught — but the interest of children in the everyday uses of percentage is easily developed and sustained. What are some of these uses?

1. School marks in the subjects of study.
2. Which class in our school had the best attendance record last month?
3. Expressing individual attendance records in per cents.
4. Keeping a graphic record of the per cent attendance of the class.
5. Batting and fielding averages of famous baseball players.
6. Standing of the teams in the big leagues.
7. Saving money by buying at reduced prices.
8. Savings effected by purchasing foods ¹ in large quantities.
9. Numerical and graphic treatment of the food elements in staple articles of diet.
10. Family budgets. For different salary levels what is the proper apportionment of income to the items of expense?
11. Miscellaneous material from newspapers and magazines such as graphic analyses, tables, etc.

In the study of topics like these it is impossible to follow strictly any logical sequence of problem types. The best the teacher can do in the way of grading difficulties is to

¹ MONROE, DAY. — "Marketing Investigations"; *Teachers College Record*, XXVI, No. 2.

foresee what problems would, in a given situation, be of most interest, and then decide whether these problems are of a suitable degree of difficulty. For example, in a certain table that has been prepared by experts in the social sciences, the per cent of income that should be apportioned to rent, clothing, food, etc., for each income level is given. The questions that naturally arise are such as these: A family has an income of \$2300. Find how much it should apportion to each item of family expense. A family with an income of \$2800 spends \$55 a month for its apartment. Is this a proper amount for it to spend? It is evident that both of these problems may be solved by finding a per cent of a number; hence this situation may be taken up early in the study of percentage. The second problem stated may be solved by finding what per cent of the income is spent for rent, and comparing the result with the norm in the table; so regarded, it is a problem of type 2.

The following project has proved interesting. The teacher's purpose in carrying it out was to teach problems of type 2 and to hold the pupils responsible for accurate work in long division.

The teacher asks the school principal to request from the class a *ready-reference table* for finding the per cent attendance of any class in the school. The class engages to do the work and the teacher promises to give the class whatever help may be needed. The following are the steps in carrying out the contract:

1. Learn how a per cent of attendance is computed.
2. Find out the enrollment of every class in the school for the purpose of finding the possible divisors (bases).

3. Find out what is the smallest attendance that is likely to occur. The numbers from this smallest attendance up to the largest enrollment are the possible dividends.

4. Prepare on squared paper the form for the table. It is better to put the possible dividends on the top line of the table and the possible divisors in the left column.

5. Assign to each row the task of finding the quotients in a given line. Continue until all the quotients have been found and checked.

6. Have each line of per cents read so that they may be copied by all the children who did not work that line.

7. Select the six best tables for use in the office and distribute the others to class teachers who are interested.

The following illustrates the form and general scheme of the table:

ATTENDANCE TABLE

ENROLLMENT	50	49	48	47	46	45	44	43
	41							
	42							
	43							—
	44						—	—
	45					—	—	—
	46				100	—	—	—
	47			100	97.9	—	—	—
	48		100	97.9	95.8	—	—	—
	49	100	98	95.9	93.9	—	—	—
	50	100	98	96	94	92	—	—

In baseball statistics the batting and fielding averages and the standings of the teams are expressed as three-place decimals, but, unfortunately, they are commonly

called *per cents*. It is well to call attention to this fact, and to teach the children to read these numbers in two ways: .462 as 462 thousandths, or as 46.2 per cent. The writer has found baseball data very interesting to boys; such data are found in publications like the World Almanac as well as in the daily newspapers. The most interesting problem is to find and compare batting averages, a problem of type 2. Of course the related problems of types 1 and 3 can be made interesting despite their relative unreality.

Every teacher who has made use of large units of work involving coöperation, requiring absolute accuracy, and leading to some well-conceived end will agree that children as a rule respond favorably to the demands of the situation for painstaking work. The writer once had a class of girls in the seventh grade who were perfectly content with a class average of 50 per cent accuracy in the fundamental operations. But these same girls, when given (in imagination, of course) a job as Pay Roll Clerk, were able to work out a time sheet, compute each man's wages, prepare the change memorandum, and hand in neat and orderly pieces of work with practically 100 per cent accuracy. Only two of the class had to take their papers back for the purpose of correcting errors, and these were not at all pleased with themselves.

EXERCISES

1. Prepare a list of exercises designed to fix in mind the meaning of *per cent*.
2. In reducing one-place decimals to per cents what principle of fractions is applied? Would you make use of this principle in teaching

the reduction in question? Give the essential steps in such a lesson and provide a few exercises for practice.

3. In reducing decimals of three or more places to per cents what principle of fractions is applied? Would you make use of this principle in teaching this reduction? Give the essential steps and provide a few exercises for practice.

4. Prepare a list of exercises to be used in teaching the per cent equivalents of fractions with denominator 8. Use the first method on page 401.

5. Prepare a list of exercises to be used in teaching the per cent equivalents of fractions with denominator 6. Use the second method on page 401.

6. Which of the above methods do you prefer? Why?

7. Prepare a 20-question test on reductions, in which the children are to work mentally and write their answers at a given signal. State the subject matter covered by the test.

8. Prepare a 20-question test on mentally finding per cents of numbers. The purpose of the test is to find what kinds of examples require further practice.

9. Prepare a drill exercise designed to bring out and fix in mind the distinction between " $\frac{3}{4}$ of" and " $\frac{3}{4}\%$ of," and the like.

10. Children often have trouble with problems of type 2 when the answer is more than 100%. Tell how you would help them with such problems.

11. Prepare a lesson, or series of lessons, of the project type; select some topic from page 405 or find a topic elsewhere.

12. State briefly the precise difference between the early meaning of *per cent* and the modern meaning. How did this change of concept, or definition, affect teaching?

CHAPTER XV

PERCENTAGE — BUSINESS APPLICATIONS

TEACHER'S KNOWLEDGE

Scope and purpose of this chapter. — So far as the mathematics is concerned there is nothing new to be taught in connection with the common business applications of percentage. In modern textbooks these applications usually include the following topics :

Profit and loss — often including the marking of prices.

Commercial discount — usually limited to the simple elements of the subject.

Commission — as applied to the activities of buying goods, selling goods, and collecting money.

The tendency is to omit all problems except those that fairly represent conditions which occur in business practice.

In order to teach these topics acceptably the teacher must have a good general knowledge of business usage ; he should be familiar with the kinds of problems that occur and the business man's methods of solving them ; and he should know the technique of teaching percentage — the methods and devices which will help the learner to realize the practical situation with which he is dealing and to perceive and understand the quantitative relations involved. The last element mentioned has been discussed quite fully in Chapter XIV ; it is the main purpose of this chapter to discuss the other elements — the business

practices, the kinds of problems, and their solution. However, we shall not attempt any exhaustive treatment of these topics. What we may properly say must be addressed to the teacher rather than to the prospective business technician; hence we must omit certain details which would be important, and include some things that would be out of place, in a textbook on commercial arithmetic.

DEVELOPMENT OF COMMERCIAL DISCOUNT

Definitions. — Manufacturers and dealers, both wholesale and retail, often sell their goods for less than their quoted prices. The deduction allowed from a regular, or quoted, price is called a *discount*. The principal reasons for allowing discounts are :

1. *Fluctuating prices.* — Some business houses issue printed catalogs or price lists. To save the expense of reprinting these catalogs every time the prices change the list prices are made higher than the actual prices are likely to go, and then the frequent changes in the market are met by quoting discounts.

2. *Large-scale buying.* — The expenses incurred in selling a large order of goods are not much greater than they are in selling a small order. The small order is likely to entail just as much correspondence, just as much of an agent's time, just as much bookkeeper's time, etc., as the large order. On this account the large-scale buyer is likely to receive a small discount in addition to the discount quoted to all buyers. Also, the dealer will make reasonable reductions to hold the trade of especially desirable customers.

3. *Prompt payments.* — In every line of business there is always more or less loss from bad debts, and also from delayed payments which require special correspondence and accounting work and sometimes necessitate the expense of forced collection. Any buyer who has an established reputation for prompt payment (whose credit is good) is justly entitled to a discount for that reason. To stimulate prompt payments it is a common practice to print on bill heads the terms of payment. For example, 3/10, N/60 means that if the bill is paid within 10 days, the payee in remitting the amount due may deduct 3% ; if it is paid after 10 days but within 60 days, the net amount of the bill is due, and it is understood, though not always enforced, that if payment is delayed beyond the 60 days, interest is chargeable for the period of such delay.

4. *Competition.* — It may be that in a certain territory there is especially active competition to meet. This involves "price cutting" by means of special discounts.

5. *Special or trade discounts.* — It has long been customary in certain retail trades to make special discounts to certain classes of people. A dressmaker, a carpenter, or a plumber can buy the materials needed for a certain job at a discount. His estimate of the cost of the job is based on the regular price of the material — which gives him an extra profit. Certain professional classes, such as clergymen and teachers, are usually given the benefit of a small discount.¹

The price printed in a catalog or price list is called the

¹ This has been explained as a survival of the special privileges accorded *clerks* (people who could read and write) during the Middle Ages. At that time people with such accomplishments were exceedingly rare ; because of their rarity, and the exceptional services they could render society, they were shown peculiar deference.

list price. A discount allowed in a commercial transaction is called a *commercial discount*; if the buyer is also a dealer (trader), the discount is called a *trade discount*; if the discount is made for cash payment, it is called a *cash discount*. The price paid for goods after the discounts have been deducted is called the *net price* — except when the terms printed on the *bill* permit the buyer to make an additional discount for prompt payment, in which case the net price is the amount of the bill before the cash discount is taken.

Technicalities. — A single discount is a per cent of the list price, or the regular price, as the case may be. But with so many different reasons and occasions for giving discounts it frequently happens that a series of two or more successive discounts occurs. This is known as a *discount series*. In computing a series of discounts the first discount is based on the list, or regular, price, the second discount is based on the remainder found by subtracting the first discount from the list price, the third discount is based on the second remainder, and so on. For example: If the list price is \$200 and discounts of 40%, 20%, and 5% are allowed, what is the net price? The first discount is 40% of \$200, or \$80, leaving a remainder of \$120; the second discount is 20% of \$120, or \$24, leaving a remainder of \$96; and the third discount is 5% of \$96, or \$4.80, leaving a net price of \$91.20.

Marking prices. — In some trades instead of preparing price lists dealers mark prices on the goods themselves. In many retail stores the selling prices are marked in plain figures, but if the cost of the article is given, this is always in code. In many lines of trade it is customary

to mark both cost and selling price in code, or cipher. Any word or phrase containing ten letters may be used as the key — the letters in order representing the figures. For example, if the following key is used :

R E P U B L I C A N
1 2 3 4 5 6 7 8 9 0

the price \$2.38 would be indicated thus: EPC. To make it harder for the buyer to decipher the code a *repeater* is often used, so called because it is used to express the same figure as the letter to its left. Thus Z might be our repeater, in which case \$2.33 would be written, EPZ.

Finding net price. — The most commonly used type of problem is that in which the net price or the purchase price is to be found. This type is an application of type 1 in percentage, and the problems are easily solved as soon as the technicalities are understood. In problems in which a series of discounts occur, different methods of solving arise which will be explained in the illustrations that follow.

Mental solutions. — A helpful *working explanation* is to call off the results of each simple problem; if a longer explanation is necessary, abbreviate the statements of each of the simple problems. To illustrate: An article listed at \$5 was sold at discounts of 25% and 20%. What was the net price? *A longer explanation:* What was the first discount? *Ans.* \$1.25. What was the remainder? \$3.75. What was the second discount? 75¢. The net price? \$3. *A briefer working explanation:* \$1.25; \$3.75; 75¢; \$3.

Two methods of written solutions. — A plumber ordered from a supply house a bill of goods amounting to \$850. The trade discounts were 30% and 20%, and the terms of payment were 2/10, N/60. Find the cash price of the goods.

FIRST METHOD

L. P., \$850	
Dis., 30% L., 20% R.	85
Cash D., 2% Net P.	3
Cash P., ?	<u>255</u>
Dis ₁ , \$255	
R ₁ , \$595	
Dis ₂ , \$119	4.76
N. P., \$476	<u>2</u>
Cash Dis., \$9.52	9.52
Cash P., \$466.48	Ans.

The scratch work shown is enough for even the less capable pupils. Many would be able to find $\frac{3}{10}$ of 850 without showing any computation at the right.

The steps in the solution are easily followed.

SECOND METHOD

L. P., \$850	$\frac{98}{100} \times \frac{4}{5} \times \frac{7}{10} \times \frac{17}{850} = \frac{392 \times 119}{100}$
Dis., 30% L., 20% R., 2% R.	
Cash P., ?	
Cash P., 98% of 80% of 70% L.	392
Cash P., \$466.48	<u>119</u>
	3528
	392
	<u>392</u>
	46648

In the first method the discounts are computed separately. It is the method used by a billing clerk in computing the net amount of a bill. In case the buyer decides to pay cash he might figure the net price in the same manner in order to check the accuracy of the bill

and then make a further deduction of 2%. Or he may use the cancellation method here illustrated.

Another method of handling a problem when there is a series of discounts is to (1) compute a single discount rate equivalent to the series, (2) multiply the list, or original, price by this single rate. Let us illustrate the procedure for step (1) by solving the problem: Find the single discount rate equivalent to the series 30% and 20%.

If the first discount is 30% of the list price, the first remainder is 70% of the list price. If the second discount is 20% of the first remainder and the first remainder is 70% of the list price, the second discount is 14% (20% of 70%) of the list price. If the first discount is 30% of the list price and the second discount is 14% of the list price, the total or single equivalent discount is 44% of the list price; hence the single discount rate is 44%. This answer may be obtained by subtracting the product of the two given rates from their sum; thus, $30\% + 20\% - 30\% \text{ of } 20\% = 30\% + 20\% - 6\% = 44\%$. By further trials the reader may satisfy himself as to the truth of the rule: *To find a single discount rate equivalent to two given discount rates, subtract the product of the given rates from their sum.*

To find a single discount equivalent to a series of three or more discounts, find the equivalent for the first two discounts, find the equivalent of the result and the third discount, etc.

Finding a discount rate. — This type of problem arises whenever it is necessary to establish or to alter prices. The following problems illustrate the situations which arise.

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1. An article is listed at \$28. The conditions of the market are such that it must be sold for about \$19.50. What rate of discount should be given?

2. An article listed at \$55 has been selling at a 20% discount. The market changes so that it is necessary to sell it for about \$37.50. What additional discount must be quoted?

PROBLEM 1

List, \$28	28.
	<u>19.50</u>
Market, \$19.50	8.50
Dis. rate, ?	.303
	<u>28</u> 8.50
Dis., \$8.50	<u>84</u>
	100
Dis. rate, 30% Ans.	<u>84</u>

PROBLEM 2

List, \$55	44.
Dis. ₁ , 20% L	<u>37.50</u>
Market, \$37.50	6.50
Dis. ₂ , ?%	
	.147
Dis. ₁ , \$11	<u>44</u> 6.50
R ₁ , \$44	<u>44</u>
Dis. ₂ , \$6.50	210
Dis. ₂ rate, 15% Ans.	<u>176</u>
	340
	<u>308</u>

Finding the list, or original price. — This type of problem seldom, if ever, occurs in practice. The following are probably as plausible as could be cited.

1. A certain New York store gives a 6% discount to teachers. Miss H remarked: "That discount saves me, on clothing alone, twenty to twenty-five dollars a year; I buy practically all my clothes there." Find about how much Miss H would spend for clothes a year if she did not get the discount.

This is a problem of type 3. The solution is left to the student.

2. A teacher who gets a 5% discount at a clothing and men's furnishing store found that his bills for the year 1924 at this store totaled \$363.55. What would have been the cost of these goods at regular prices? How much did he save?

This is a problem of type 5. The solution is left to the reader.

No problem of type 4 could arise in connection with commercial discount, because the base is never subject to a per cent increase.

DEVELOPMENT OF COMMISSION

Definitions. — Of course, the majority of men and women engaged in commercial pursuits are not proprietors, but are in the employ of business firms and corporations. In the general sense that they are *acting for others*, people so employed are *agents*. However, the term *agent* is commonly used in a somewhat narrower sense, to denote (1) a person who buys goods for another, (2) a person who sells goods for another, or (3) a person who collects money for another. The remuneration received for buying, selling, or collecting for another is called *commission*.

One who buys goods for another is called a *purchasing agent*, a *buyer*, a *broker*, or a *commission merchant*,¹ the term used varying with the kind of business and the precise nature of the service rendered. In many cases such an agent has no other function than that of buying goods; often he has other duties to perform — for example, he may be primarily responsible for the management of a department in a large store. A good purchasing agent must have expert knowledge of the goods he buys; he must know the various sources of supply, be well informed about prices, and be able to judge of the quality; he must possess the knowledge and experience that will enable him to buy in such amounts and of such varieties and at such prices as will most likely meet the demand. Thus,

¹ It is common to distinguish between broker and commission merchant; the former merely arranges the transaction while the latter has actual possession of the goods. There are, however, exceptions.

we have cotton experts, silk experts, steel experts, etc. Such people render a very important service, for if the agent buys unwisely in any respect, a loss is sure to result, and in the long run such losses are sustained by the consumer.

The term *broker* applies alike to one who buys and one who sells. The broker is not so likely to be an expert in the selection of goods; his function is rather to manage the business aspects of buying or selling, leaving the technical work, if any is required, to experts whom he employs.

One who sells goods for another is called a *salesman* (*saleswoman*, *salesperson*), *traveling salesman*, *drummer*, *commission merchant*, *agent*, or *broker*, the term used depending upon the nature of the service rendered. The *salesman* may sell over the counter or on the road, but the commercial traveler or drummer always calls on his prospective customers. A good salesman must understand the psychology of salesmanship — how to get and hold the attention of the customer — he must be apt to see what his customer needs and meet that need to the best of his ability. A good salesman understands that a business transaction must be a benefit to the buyer as well as to the seller; he will not try to sell an article unsuited to the needs of the buyer.

Different kinds of agents are paid for their services in different ways. A purchasing agent or buyer is usually paid a regular salary and a commission based sometimes upon the quantity of goods bought, sometimes on their cost; or, if the buyer is the responsible head of a department, his commission may be based upon the sales, or

upon the net profits, of the department. The plan of remuneration is so designed as to stimulate the best efforts of the agent. An *inside* salesperson (one working in the store or showroom) receives a regular salary and usually a relatively small commission. The traveling salesman or drummer may work on a purely commission basis, or upon a salary and commission. His traveling expenses may be paid by his employer, or by himself.

Selling through commission merchants. — Mr. Wm. T. Simpson, who has a poultry farm near Johnstown, N. Y., had a large number of broilers he wished to sell. There being no demand near home, he shipped the broilers in crates to Smith and Tanner, commission merchants, in New York City. He wrote them a letter asking them to sell the chickens at the best price obtainable. Smith and Tanner sold them at 32 cents a pound live weight, the total weight being 386 pounds. Smith and Tanner charged Mr. Simpson 25¢ a day per crate for storing the 8 crates for the two days intervening before they were sold; they also charged \$2 for drayage, and a 5% commission.

Smith and Tanner then sent Mr. Simpson an *account sales* like that on the following page, and inclosed therewith a check for the net proceeds.

The transaction described above illustrates the function of the commission merchant. Formerly a large amount of business was done in this manner, but many kinds of produce that used to be sold through commission merchants are now coming to be sold through agents employed by syndicates, or associations, of producers. As the producer exercises increasing control of the marketing

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of his goods the business of commission merchants decreases. Here is the *account sales* involved.

ACCOUNT SALES

NEW YORK CITY. Aug. 27, 1924

Sold for the account of

WM. T. SIMPSON, JOHNSTOWN, N. Y.

By SMITH & TANNER

Commission Merchants

Aug.	25	190 Broilers, 386 lb.	32¢			123	52
		Charges					
Aug.	23	Drayage		2			
Aug.	25	Storage, 8 crates, 2 days.		4			
		Commission, 5%		6	18		
		Net Proceeds		111	34		
				123	52	123	52

In the transaction we have described the chickens were the *consignment*, Mr. Simpson was the *consignor*, and Smith & Tanner the *consignee*. The selling price, \$123.52, was the *gross proceeds* and the amount of the check sent to Simpson, \$111.34, the *net proceeds*.

Buying through commission merchants. — Commission merchants engage also in buying activities. The amount paid for the goods is called the *prime cost*. The entire cost, including commission and other expenses, is called the *gross cost*. The statement which the commission merchant renders his principal is called an *account purchase*.

The following illustrates the account purchase. Note that in the account sales the selling price of the broilers is entered in the right-hand (credit) column, and that the charges and net proceeds appear in the left-hand

(debit) column. In the account purchase the cost of the goods and the charges must appear in the debit column — that is, Mr. Backus is indebted to Connolly & Co. for *all* these items. In case the principal remits with his order, to establish a credit, the amount of the remittance would appear in the credit column.

ACCOUNT PURCHASE

DULUTH, MINN., Aug. 18, 1924

Purchase for the account of

LESTER J. BACKUS, PALMYRA, N. Y.

By FRANCIS J. CONNOLLY & Co.

Lumber Merchants

40 M Spruce Lumber	\$38.00	1520				
25 M White Pine Boards	46.50	1162	50			
60 M Cedar Shingles	9.25	555				
15 M Oak Flooring	43.75	656	25	3893	75	
Charges						
Cartage, \$2 per M		280				
Commission, 2½%		97	34	377	34	
Amount charged to your acc't.				4271	09	

Collecting accounts. — When business houses find it especially difficult to collect certain bills, they usually place them in the hands of a collection agency. Because of its organization such an agency is able to collect a very considerable proportion of doubtful, or slow, accounts, and therefore to render a valuable service.

One such agency has in cities and villages all through the eastern states sub-agents called *bonded attorneys* who take care of the details of collecting. Suppose, for example, that a restaurant equipment house in New York

City has sold on a monthly payment plan \$850 worth of equipment to a restaurant in Trenton, N. J. Payments are not made when due, and the creditor finds that he is not accomplishing anything through correspondence. He decides to send the account to this agency's New York office for collection. This office forwards the account to its bonded attorney in Trenton, who does his best to collect it.

Whenever possible the attorney will avoid entering suit in the courts; but if the amount is large, and the chances of forcing payment are good, the debtor will be sued.

The following excerpts from the terms of agreement between the company and its bonded attorneys show the commission arrangements — what rates of commission are received for collections in various amounts.

1. Without Suit. — 15% on the first \$300.00 or less collected.
 8% on the next \$700 or less collected.
 4% on the balance collected in excess of \$1000.00.
 Minimum fee, \$5.00.

When the collection is made by suit, the commissions are the same as above, except that the total cost of collection shall not exceed half the claim. The rules governing collection of claims by suit are too complicated to be stated in full here.

If the above account was collected, the creditor received \$850 — \$45 (15% of \$300) — \$44 (8% of \$550), or \$761.

DEVELOPMENT OF PROFIT AND LOSS

Definitions. — The term *profit*, taken without modifiers, has no precise meaning. It is true that the difference between the cost of an article and its selling price is the

profit. But what do we mean by *cost* and what by *selling price*?

Take the simple case of the man who goes about the country buying blueberries to ship to the city market. If he pays 18¢ a quart and sells them at 30¢ a quart, he is apparently making a profit of 12¢ a quart; but there are certain items of expense incurred, and his time is worth something.

The difference between *cost price* and the *selling price* is called the *gross trading profit*. In the case cited the cost price is 18¢, the selling price is 30¢, and the gross trading profit is 12¢ per quart.

The term *cost price* is not to be confused with the term *cost*. The *cost price* of an article is the amount of money actually paid for it, while the cost in many instances will include also the expenses incurred in the process of buying. If the expenses of travel and the man's ordinary earning capacity when working for somebody else amount to 2¢ a quart on the berries handled, the cost per quart, when the product is ready to sell, is 20¢.

Likewise there are expenses incurred in selling. This man has to pay express on the crates of berries and has to bear losses due to wear and tear on the crates. Occasionally the berries will not arrive at their destination in marketable condition, in which case they bring a reduced price or nothing at all. When these items have been deducted from the selling price, the remainder is called the *net selling price*.

The *net profit* is the difference between the cost and the net selling price; or it may be thought of as the gross trading profit minus all business expenses.

The business situation cited above is a very simple one. It was chosen for that reason. But it illustrates the principles that apply in a large business where the figuring of costs and profits is very complicated, requiring the services of experts in such matters.

Let us consider the business of a large department store. Every department is expected to yield a net profit, and if it fails to do so, some change in its size, its methods, the varieties of goods shown, or its personnel is likely to be made. This makes it necessary to keep account of profits and losses by departments. Such items as salaries and buying expenses are called *local department expenses*. Then there are the general expenses of the business as a whole, such as rent, interest on loans, insurance, heat, light, janitorial and elevator services, advertising, and administrative and accounting expenses. All such items are classed as *overhead expenses*.

Since overhead expenses are not directly incurred by the departments, but are for the benefit of all departments in varying degrees, it is necessary to distribute these expenses among the departments according to certain reasonable rules; such distribution is called *prorating*. For example, the item of rent would be *prorated* according to the floor space occupied by the departments — probably charging more for space on the street floor.

Turnovers. — Many firms find it advantageous to offer goods for sale at lowered prices with a view to selling out the stock as quickly as possible, then buying up a new stock of the same goods and selling them in the quickest possible time, and so on. These frequent *turnovers* of stock, as they are called, often yield greater profits at

the end of a given time than would have accrued by a slow sale of the stock at a higher price. The overhead expenses for a given period of time will not be very much greater with several turnovers of stock than with one turnover. Illustration: Suppose a firm marks up goods that cost \$20,000 to 15% above cost and at this price makes four turnovers of stock. If the overhead expenses for the time amounted to \$4000, did the firm net more or less than by selling the goods on slow sale at 50% above cost? The total cost for the four turnovers was \$80,000; the selling price of the stock at 115% of the cost was \$92,000; the gross profit was \$12,000; the net profit was \$8000. At 150% of the cost the selling price would have been \$30,000; the gross profit \$10,000; the net profit, \$6000. Hence the four turnovers netted \$2000 more.

Per cent profit. — “ A questionnaire¹ recently sent out to representatives of business houses, containing a question as to the value on which the rate of profit or loss is reckoned, brought the following information :

Number taking selling price as base	96
Number taking first cost plus added costs	45
Number taking first cost	24
Number using a combination of above	27
Number of replies received	<u>192</u> ”

It appears that one-half of these business houses used the selling price as base, about one-quarter used cost price plus buying expenses, and the remaining quarter split rather evenly between cost price and a combination base.

It is quite evident from the above that the practice of

¹ HART, C. and WATTS, M. J. — *Commercial and Industrial Arithmetic*, p. 120; D. Appleton and Company.

reckoning the profit or loss upon the cost is no longer a general one. Some elementary texts still persist in teaching the old practice, but all the recent commercial texts explain the customary business usages and base much of their problem material upon profits reckoned on the selling price.

Solution of problems. — The technique of the solution of problems in profit and loss differs not at all from that described in the chapter on percentage (pp. 391–397).

The types of problems that occur in profit and loss are very numerous — some of them extremely technical. All of the five type cases are represented. The first step necessary in understanding the situation is to know upon what the profit or loss is reckoned. Otherwise the general procedure is as described in the solutions for the five types in percentage. Much drill on solving mental problems should precede written solutions. A short working explanation is helpful. Illustration: Goods were sold for \$70 at a gain of 40% of the cost. What was the gain? *Explanation:* What part of the cost was the selling price? *Ans.*, 140% C. or $\frac{7}{5}$ C. What is $\frac{1}{5}$ of the cost? *Ans.*, \$10. What is the gain or $\frac{2}{5}$ of the cost? *Ans.*, \$20. *Short working explanation:* S.P., $\frac{7}{5}$ C.; $\frac{1}{5}$ C., \$10; $\frac{2}{5}$ C. or gain, \$20, *Ans.*

Graphic aids are especially valuable in problems involving cases 3, 4, and 5. Illustration: When goods sell at 20% above cost, what per cent of the selling price is gained?

G. is $\frac{1}{5}$ of S.P. or $16\frac{2}{3}\%$ S.P. *Ans.*

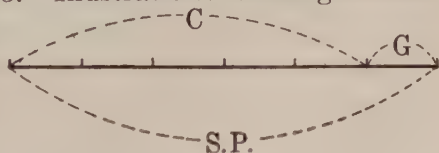


FIGURE 93

EXERCISES

1. One year the King Car Company made the following proposition in an advertisement: "If by August 1 we have sold 500,000 cars this season we will refund to each buyer 13% of the cost of his car." Supposing that 547,438 cars were sold by Aug. 1, find Mr. A's refund on a car costing \$2850. Find the amount refunded by the company if the average price of the cars was \$2650.

Explain the purpose of an offer of this sort.

2. Copy and complete the following bill and find how much H. C. Robbins must remit if he pays August 10. The placing of the items is indicated by the question marks.

GRAND RAPIDS, MICH., *Aug. 5, 1924*

Mr. H. C. ROBBINS,
AUBURN, N. Y.

Bought of EUREKA FURNITURE COMPANY

Terms: 2%, 10 days; net, 60 days.

8	Flat Top Desks	\$45	?		
12	Dining Tables, 48"	38	?		
9	Dining Tables, 42"	35	?		
96	Dining Chairs	7.50	?		
18	Tea Carts	16.75	?		
			?		
	Less 25% and 12%		?		?

3. Today you are shipping James C. Bragdon of Fort Plain, N. Y., the following items of hardware: 3 dozen chisels at \$5.60 a dozen, 4 dozen handsaws at \$9.75 a dozen, 6 dozen hammers at \$6.65 a dozen, 800 lb. nails at $3\frac{1}{4}$ cents a pound. On the tools there are trade discounts of 15% and 5%, and on the nails a discount of 12%. Terms: Net 60 days, 3% 10 days. Make out the bill for the shipment.

NOTE. — After the tools have been put on the bill, make the discounts and enter the net amount in the right-hand column, as in Exercise 2.

Then proceed in the same way with the nails. Now total the two items in the right-hand column.

4. Using the key **R E P U B L I C A N** as given on page 414, solve the following problem: The original price of a dress was marked at **ELBN**. It was first marked down 20% and later a second markdown of 25% was made. What price did it bring at the special sale after the second markdown?

5. An agent receives a regular salary of \$120 a month and 4% commission on his sales; his traveling expenses are paid by his employer. Find his average earnings per month if his sales for the year amounted to \$137,430.

6. Mr. Ward is a traveling salesman. He gets a regular salary of \$120 a month and a 4% commission. He says that he earned \$3708 last year. Find the amount of his sales.

7. A young man worked during his summer vacation selling aluminum ware. In the twelve weeks he sold \$1368 worth of goods. His expenses averaged \$18 a week. What were his net earnings if his commission was 40%?

8. A saleswoman in a department receives a regular salary of \$18 a week. If her sales for a week exceed \$500, she is given a 3% commission on the excess. During the first three weeks in December her sales totaled, the first week \$879.50, the second week \$1057.38, the third week \$1246.74. Find her total earnings for the three weeks of the holiday rush.

9. An auctioneer charges 5% for his services, with the understanding that his minimum charge is \$10 a day. He was engaged to auction off Mr. Andrews's household goods. If the sales for the day amounted to \$1275, what did the auctioneer earn?

10. What is the least amount of sales that will yield to the auctioneer his minimum charge at 5%?

11. The head of the silk department in a certain store does all the buying for his department. He receives a salary of \$200 a month and 2% of the net profits of his department. Last year his total income was \$4650. What were the net profits of his department?

12. Real estate agents in and about New York City charge a commission of 5% on sales and rentals of real estate, except that the commission for summer rentals of suburban houses is 10%. Find the total

earnings of an agent who does the following business during the year: Total sales, \$143,600; long-term rentals \$12,470; summer rentals, \$3465.

13. Prepare an account sales from the following data, using your own name as agent, and any name you choose as principal:

On October 15 you receive a consignment of 200 cases of eggs (30 doz. in a case). On the same day you pay a truckman \$20 for bringing the eggs from the freight station to your warehouse, and insure them at a cost of \$4.50. On October 18 you sold 50 cases at \$10.50 a case and 35 cases at \$10.75. On October 20 you sold 60 cases at \$10.35, and the next day you sold the balance at \$10.60. You charge \$15 for storage and a $2\frac{1}{2}\%$ commission.

14. Write a letter and the check which should accompany the above account sales.

15. The following is a statement of the collections made by a bonded attorney during the month of July, 1925. Copy, fill in the blank columns, and find the total commissions and remittances. Check your work. Use rates quoted on page 423.

COLLECTION BUSINESS, JULY, 1925

CREDITOR	DEBTOR	COLLECTED	COMMISSION	REMITTED
J. Swan	L. Snow	250.00		
H. Blatt	R. Romm	89.65		
F. G. Starr	B. Silver	2976.		
M. J. Hare	G. K. Ford			835.72
D. E. Pratt	L. J. Tinney		116.04	

16. A fruit grower planted 10 acres of land to peach trees, 120 trees to an acre. The land is valued at \$425 per acre; the trees cost \$15 per hundred; preparing the ground and planting the trees cost \$25 per acre; the yearly taxes were \$42; care of the orchard for the first four years cost \$260. Find the total cost of the orchard at the end of the fourth year.

17. The orchard begins to bear fruit the fifth year. 225 baskets of peaches were sold at 95¢ a basket. The baskets cost \$8 per hundred;

the picking and sorting cost 11¢ a basket; selling expenses were \$35; care of orchard \$90.

(a) Find the net profit, or loss, for the year.

(b) What per cent return did he get on the total investment?

NOTE. — The answer to problem 16 is the amount of money needed in bringing the orchard to a point where it begins to produce an income; this we think of as investment. The expenditures made during the fifth, and subsequent, years do not constitute any additional capital outlay or investment, because they can be taken out of income; such expenditures are called operating or production expenses.

18. The next year the orchard yielded 675 baskets of fruit which sold at \$1.10 a basket. Reckoning the expenses as in problem 17, find net profit and per cent return as before.

19. The following table is a compact statement of the profits and losses in the departments of a business concern. The word *inventory* as here used is the cost of the goods in stock on the date named. In figuring the profit, or loss, it is necessary to take into account the change in inventory from the beginning to the end of the year.

Copy and complete the table,¹ using the double check (p. 37, Ex. 12).

INVENTORY JAN. 1, 19—		PURCHASES		BUYING EXPENSES		TOTAL COST OF PURCHASES		INVENTORY DEC. 31, 19—		COST OF GOODS SOLD		SALES		PROFIT OR LOSS		PER CENT PROFIT OR LOSS	
3125	75	16,294	90	370	25			4685	60			18,495	40				
5428	80	23,926	56	627	90			5626	35			33,756	20				
1290	00	6,754	00	271	25			2140	00			8,325	75				
4728	50	12,398	80	1417	25			1265	70			13,215	75				
3685	00	15,927	00	485	00			2726	00			14,850	00				

20. A retailer bought hats at \$45 per dozen and sold them at \$5.50 apiece. Find the per cent gain (a) based on the cost, (b) based on the selling price.

¹ FINNEY and BROWN. — *Modern Business Arithmetic*, p. 256; Henry Holt and Company.

21. A furniture store bought an invoice of dining-room furniture at discounts of 30% and 10% and sold it at list price. What per cent of the cost was gained? What per cent of the selling price was gained? Show the relations by a line diagram, and the formal written solution. Explain the problem by giving the simple problems and answers.

22. Buttons purchased at 48¢ per gross were sold so as to gain 125% of the cost. Find the selling price per dozen. Solve mentally, stating the simple problems and answers.

23. If the gain is 75% of the cost, what per cent of the selling price is it?

24. Bought pianos at a 40% discount from list and sold them at 15% discount from list. The profit was what per cent (a) of the cost, (b) of the selling price? Show relations with a line diagram.

25. What particular kind of profit are we dealing with in problems 20 to 24?

26. Find the value of the missing terms, using mental computation so far as you can.

COST	SELLING PRICE	GROSS TRADING PROFIT	PER CENT PROFIT ON COST	PER CENT PROFIT ON SELLING PRICE
\$50	\$75			
30	40			
25		\$15		
20			25%	
35				30%
	80	20		
	120			16 $\frac{2}{3}$ %
	50		66 $\frac{2}{3}$ %	
		125	25%	
		25		20%

METHODS OF TEACHING

Commercial discount. — Probably the best way to introduce the idea of a discount is through the use of

advertisements of cut-price sales. Solutions of some problems based upon such material will give a mastery of the number relations involved, thus preparing the pupil for a study of the other reasons for allowing discounts. The following is a feasible order for the study of these reasons: (1) fluctuating prices making necessary the quotation of discounts from list, or catalog, prices, (2) prompt payments, (3) large-scale buying. Children should be encouraged to bring to class catalogs and to make up and solve problems based on the catalog data. It is customary in this connection to review the subject of bills and to introduce the procedure of discounting bills.

After the children have figured single discounts for these various reasons, the computation of discounts in series is introduced. It will be easy for the children to understand that the net amount of a bill is the remainder after the discount on the list price has been deducted, and that a discount for prompt payment would naturally be applied to the net amount of the bill.

The finding of a single discount equivalent to a discount series is quite difficult, and is not of sufficient importance to be taught to children.

Commission. — The reasons for the procedures in situations involving commission and the meaning of the technical terms can be best taught through some specific situation. It is best to begin with a situation which comes close to the experience of the children. In rural communities the work would center around the marketing of produce. Some of the children would know that their fathers ship chickens, veal calves, fruit, or vegetables to

the city to be sold ; and if possible, the problems should grow out of some of these real situations.

In urban communities where the children live in tenements or apartment houses, the best approach might be through the study of the work of renting agents. In sections where the people own their homes it is probable that many of the parents are working on a commission basis or come in contact with business done on that basis. It is frequently difficult to obtain from business men specific information about their business, but if the children are urged to ask questions they are sure to obtain some information that can be used in class work.

Commission is usually studied in the seventh year. When it is begun in the sixth year, it is well to confine the work of that year to the study of situations of local interest and to problems arising therefrom.

It is neither wise nor necessary to take up all the applications of commission in a common school course. It is sufficient that the children learn the general principles as they apply to those situations that are most interesting and important to them. The principal reason for taking up so many applications in this chapter is that the teacher who knows the subject in a broad sense, and who has a realization of its social significance, will be better able to take it up at the right end and in a manner that will enlist the interest of his pupils. Let us suggest at this point the possibility of correlating commission with the study of our food supply and the manner of its distribution — the problem of getting food from the producer to the consumer in the most economical way.

In general the children should be required to use formal

written solutions of problems; however, if *account sales* and *account purchase* are studied, the solutions will take the form shown on pages 421 and 422. The character of the formal written solution has been so frequently illustrated in this and the preceding chapters that it will be unnecessary to illustrate here. Problems of types 3, 4, and 5, if taught at all, might be deferred until the children understand the use of the algebraic equation.

Profit and loss. — It is usual to begin with simple problems in finding the amount of profit or loss, and follow immediately with problems in finding the per cent of profit or loss. Oral solutions will precede written solutions. There is a distinct advantage in making these early problems connect as closely as possible with the experience or observation of the pupils; if this is done, the textbook problems will take on more meaning.

Practically all real problems in profit and loss are of the first two types; if the other types are taught at all, they might be deferred until the children have learned the use of the equation.

Most of our elementary textbooks still state that the per cent gain or loss is always based on the cost. It is very easy to learn this, and the writers have rarely found a high school graduate who didn't believe it. Evidently it is not true. It would be very much more difficult to teach the whole truth about the matter — probably so difficult that it would be unwise to attempt anything like a complete treatment of the subject below the high school. In the writers' opinion the best plan is to teach that there is no uniformity of practice in actual business, but that very many business houses figure their profit as a

per cent of the selling price. The teacher who knows something about the procedures practiced in business will be able to select from the textbook the children are using such problems as fairly represent business practice and to supplement the text with problems of his own finding or making. In general such problems should definitely state what is to be used as the base.

EXERCISES

1. Procure from some jobber or wholesale house a descriptive catalog, or a price list, which is subject to discounts. Tell how you would use this price list in teaching discounts allowed because of fluctuations in prices.

2. Tell what kind of material you would use and how you would proceed in teaching discounts on account of large-scale buying.

3. If you were in some retail business which would you prefer — to buy your goods from a firm that is strict in its credit policy, or from one that is lax in this respect? Give reasons. Make a series of problems which will bring out the importance of honesty and promptness in business, (a) to the buyer, (b) to the seller, (c) to the general public.

4. Plan a lesson on the question: If a list price is subject to a series of two or more discounts, does a change in the order in which the discounts are taken make any difference in the net price? Suggestion: Find the answer by the inductive method, and then explain why the answer must be true.

5. Plan a lesson on marking prices.

6. Prepare an outline plan showing in their teaching order the topics, or lessons, into which the subject of commercial discount should be divided. State the precise aim in each lesson.

7. Assuming that it is the first lesson on the subject of commission, prepare a somewhat detailed outline on the topic: How people sell their houses through agents. In what kind of a community would this be a suitable first lesson? What activities of agents (or what practical questions) could appropriately be taken as the subjects of lessons following the one you have planned?

8. Work out in detail a first lesson on commission for a class in a rural school 100 miles or more from any large city. Give in their teaching order a series of lesson topics to follow the first lesson.

9. The children are having trouble over the question: Shall we add or subtract the commission? Explain how you would help them, and give some problems for practice. What would you have them do with these problems?

10. Make, or find, two plausible and interesting problems of type 2 in commission; two of type 3; two of type 4; two of type 5. Using appropriate objective aids help children to solve one problem of each type (*a*) by the method of analysis, (*b*) by the equation method.

11. Prepare a drill lesson on (*a*) the terms used in commission, (*b*) mental solution of problems of types 1, 2, 3.

12. Prepare a test on some one topic in commission.

13. Prepare a test on commission as a whole.

14. Plan a review lesson on the whole subject of commission.

15. Select from Exercises 16–26, pp. 430–432, those you think would be suited to pupils in the 7th and 8th grades. Arrange these problems in order of difficulty.

16. In which of the problems selected will a line diagram be especially helpful?

17. Take some textbook commonly used in your locality, and read the author's presentation of profit and loss and the problems for practice. Work out a series of lessons on the subject using the textbook so far as you can and supplementing with such other material as you think advantageous.

18. Plan in detail the first lesson on profit and loss. Make an assignment of home work.

19. Prepare a list of exercises for oral drill suited to children who are taking their first lessons in profit and loss.

CHAPTER XVI

INTEREST

TEACHER'S KNOWLEDGE

SIMPLE INTEREST

Terms. — The borrowing of money in the business world is a common practice and is indulged in by rich and poor alike. It requires as much business acumen to borrow money wisely as it does to make investments or to carry on any financial enterprise with success. For the use of borrowed money it is customary to pay a stipulated sum of money known as *interest*. The sum of money upon which interest is computed is called the *principal*. The interest plus the principal is known as the *amount*. The borrower is usually required to give as *security* to the lender some form of negotiable paper¹ such as a certificate of stock, a bond, or a mortgage on property. Sometimes his personal note is accepted as sufficient security.

Technicalities. — Interest is reckoned by taking a certain per cent of the principal. A new element, not found in other percentage applications, enters into interest computations, namely, the *time* for which a loan is made. In long-time loans interest is due at stated intervals — annually, semi-annually, or quarterly ; in short-time loans the interest is due at the expiration of the time. The

¹ See Chapter XVII on Commercial Paper.

stipulated rate of interest is understood to be the *annual rate*; the date on which a loan falls due is called the *date of maturity*. Thus, if Mr. A borrowed \$400 on January 1, 1925, for 3 years at 6%, the principal is \$400, the interest due each year is 6% of \$400, or \$24. The interest for the 3 years will be \$72. The date of maturity is Jan. 1, 1928. The amount at maturity is \$400 plus \$72, or \$472.

The common practice in interest computations is to count 360 days as a year. In some government and bank transactions it is customary to count 365 days to a year. Interest thus reckoned is called *exact* or *accurate interest*.

Interest may be computed upon the principal only, or upon the principal and accumulated interests not paid when due. The simplest and most commonly used form is that interest which is computed on the principal only. It is known as *simple interest*.

The maximum rate of interest is fixed by law and is called the *legal rate*; if a higher rate is charged, such illegal rate is called *usury*. The legal rate in most states is 6%, though in some states the maximum is 12%. By written agreement of the parties concerned a rate higher than the legal rate is allowed in most of the states. Thus, New York has legalized any rate of interest on *call* loans¹ of \$5000 or upward on collateral security.

Several different methods of computing interest are in use, each of which has its advantages and disadvantages. It is well for the teacher to understand the principle upon which each method is based and to know the most economical procedure in applying these principles. He is then in a position to weigh the relative merits of the different

¹ A call loan is payable when the lender calls for the money.

methods and to choose that one best suited to his and his pupils' needs. Explanations and illustrations of several different methods, all of which are taught, some in one school, some in another, are given in the paragraphs that follow.

The cancellation method. — By this method no rule is necessary, although the method is sometimes taught by expressing the relations in formula form. The method is based upon the *interest for 1 day*; the total time is reduced to days and the interest for the total number of days is easily found from 1 day's interest. No computation is made until all the operations have been expressed; the result is then found by cancellation. Illustration: Find the interest of \$800 for 3 yr. 11 mo. 22 da. at 8%.

$P, \$800$		$800 \times \frac{8}{100} \times \frac{1432}{360}$	$\frac{1432}{8}$
$T, 3 \text{ yr. } 11 \text{ mo. } 22 \text{ da.}$			<u>11456</u>
$R, 8\%$		45	
$I, ?$			
$T, 1432 \text{ days}$	360	254.577	
$I, \$254.58 \text{ Ans.}$	<u>3</u>	45)11456.000	
	1080	90	
	330	<u>245</u>	
	22	225	
	<u>1432</u>	<u>206</u>	
		180	
		<u>260</u>	
		225	
		<u>350</u>	
		315	
		<u>350</u>	

8% of \$800 is the interest for 1 year at 8%; we write $800 \times \frac{8}{100}$. The interest for 1 day is $\frac{1}{360}$ of this amount and for 1432 days is $\frac{1432}{360}$ of the interest for 1 year; we write $800 \times \frac{8}{100} \times \frac{1432}{360}$. The formula

is: $I = PRT$, in which I stands for interest, P for principal, R for rate, and T for time in years.

The cancellation method is easy to understand, and can be used as a basis for developing other methods. If there is but little cancellation, a good deal of computation is necessary. The method is therefore somewhat clumsy and long and is not the best to use as a general method.

The aliquot¹ parts method. — This method is based upon the *interest for 1 year*. The months are broken up into easy fractional parts of a year (or of a preceding number of months) and the days in the same way into easy fractional parts of a month. The interest is computed for each part, and the total interest is then found by adding the parts. Illustration: Let us work the above problem by this method.

P , \$800	1 yr.	64.00
T , 3 yr. 11 mo. 22 da.	2 yr.	128.00
R , 8%	6 mo.	32.00
I , ?	4 mo.	21.333
	1 mo.	5.333
	15 da.	2.666
	6 da.	1.066
	1 da.	.177
I , \$254.58. <i>Ans.</i>		254.575

The interest is first found for 1 year, then for 2 more years. The 11 months are broken up into 6 months ($\frac{1}{2}$ of a year), 4 months ($\frac{1}{3}$ of a year) and 1 month ($\frac{1}{4}$ of 4 months); the 21 days into 15 days ($\frac{1}{2}$ of a month), 6 days ($\frac{1}{3}$ of a month), and 1 day ($\frac{1}{6}$ of 6 days). Any other convenient parts might be used so long as unit fractional parts only are used. Thus, for finding the interest for 22 days we might use 10 days ($\frac{1}{3}$ of a month), 10 days, and 2 days ($\frac{1}{5}$ of 10 days).

¹ An aliquot part is a part expressed by a fraction whose numerator is 1.

This method is a convenient method to use for periods of time in excess of a year; the only computations are the easy divisions and addition, including a possible multiplication for the additional number of years. No rule is necessary; no formula is needed; the procedure is rational; the computation work is concise and neat in form. It is also the basis for mental computations in interest. The chief disadvantage is the difficulty of breaking up the time accurately, and into no other than *unit* fractional parts. It is one of the best general methods.

The six-per-cent method.¹ — It is convenient in this method to memorize a rule, but the rule is easily obtained and if forgotten can be again worked out. This method is based upon the *interest on \$1 at 6%*. Computing the interest on \$1 for 1 year, for 1 month, for 1 day at 6%, we get the following rule, which should be memorized:

The interest on \$1 at 6% for 1 year is 6¢; for 1 month, $\frac{1}{2}$ ¢; for 1 day, $\frac{1}{6}$ of a mill.

Let us now apply this rule to the problem used in preceding sections.

P, \$800	.18	
T, 3 yr. 11 mo. 22 da.	.055	
R, 8%	.003 $\frac{1}{2}$	
I, ?	.238 $\frac{1}{2}$	
		800
I, \$1 at 6%, \$.238 $\frac{1}{2}$	533	
I, \$800 at 6%, \$190.933	190 4	
I, \$254.58 Ans.	190.933	@ 6%
	63.644	@ 2%
		254.577

¹ This is by no means the only six-per-cent method, though by established usage it is called *the* six-per-cent method. Any method by which the *interest at six per cent is first found* and a correction made for any other rate is a six-per-cent method. In this chapter three six-per-cent methods are treated.

The interest on \$1 for 1 year is 6¢, for 3 years 18¢; we write .18. The interest for 1 month is $\frac{1}{2}$ ¢, for 11 months is 5 $\frac{1}{2}$ ¢; we write .055. The interest for 1 day is $\frac{1}{3}$ of a mill, for 22 days is 3 $\frac{2}{3}$ mills; we write .003 $\frac{2}{3}$. Adding these amounts, we have the interest on \$1 for the given time at 6%. We then find the interest on \$800 at 6%. The interest at 8% is easily found by increasing the interest at 6% by $\frac{1}{3}$ of itself.

This method is concise, neat, and satisfactory as a general method. Having found the interest at 6%, it is easy to modify it for other rates; for 7%, increase the interest by $\frac{1}{6}$ of itself; for 5%, decrease by $\frac{1}{6}$ of itself; for 9%, increase by $\frac{1}{2}$ of itself; for 10%, take $\frac{1}{6}$ of the interest at 6% and multiply the result by 10 by moving the decimal point. Other rates can easily be worked out according to the same plan.

One of the disadvantages of this method is that the interest on \$1 is often in the form of a mixed number and there is sometimes a temptation to drop the fraction or to reduce it to a decimal. This will affect the answer unless the decimal is carried out to so many places as to make the decimal multiplication more difficult than the fractional multiplication. Thus in the above we should need to carry out the decimal to the 7th place before multiplying.

The thirty-six-per-cent method. — This method, too, is dependent upon a rule, which should be discovered by the learner and memorized. Let us find the interest at 36% on any principal for any number of days as we would in the cancellation method, using P for our principal and N for the number of days.

$$I = P \times \frac{36}{100} \times \frac{N}{\frac{360}{10}} = \frac{P \times N}{1000}$$

That is, we must multiply the principal by the number of days, and divide the result by 1000. It is perhaps better to divide by 1000 first, and then multiply. Since this is a mere matter of moving the decimal point, the rule is usually stated as follows :

To find the interest at 36%, move the decimal point three places to the left in the principal and multiply by the number of days.

Applying this rule to our problem, we have :

P, \$800	1080	1145.600	36%
T, 3 yr. 11 mo. 22 da.	330	127.288	4%
R, 8%	22	254.576	8%
I, ?	<u>1432</u>		
	<u>.800</u>		
I, \$254.58 Ans.	1145.600		

We first find the time in days. Moving the decimal point 3 places to the left and multiplying, we obtain the interest at 36%. We take $\frac{1}{3}$ of this, which gives us the interest at 4%. This result is multiplied by 2, which gives us the interest at 8%.

The advocates of this method like it chiefly because of the fact that from 36% it is easy to find the interest at other rates ; 9% is $\frac{1}{4}$ of 36%, 4% is $\frac{1}{9}$ of it, 6% is $\frac{1}{6}$ of it, 3% is $\frac{1}{12}$ of it, and so on. After the time is reduced to days, only one computation is necessary to find the interest at 36%. The rule is short and easy of application. The disadvantages are that the time must always be reduced to days ; if the principal consists of a number of dollars and cents, the multiplication resulting is long and clumsy ; 5% and 7%, rates frequently used, are not easy fractional parts of 36% ; and finally the result must always be modified, as 36% is a rate never used.

The six-day method. — The procedure is the same as in the thirty-six-per-cent method, except that the rate used is six per cent.

$$I = P \times \frac{6}{100} \times \frac{N}{\frac{360}{60}} = \frac{N \times P}{6 \times 1000}.$$

Hence our rule reads: To find the interest at six per cent, move the decimal point three places to the left in the principal, multiply by the number of days and divide the result by 6. The solution of our problem follows:

P, \$800	1080
T, 3 yr. 11 mo. 22 da.	330
R, 8%	22
I, ?	<u>1432</u>
	.800
I, \$254.58 <i>Ans.</i>	6 <u>1145.600</u>
	190.933 @ 6%
	63.644 @ 2%
	<u>254.577</u>

The explanation is similar to that for the thirty-six-per-cent method. The steps are in direct application of the rule.

NOTE. — If preferred, the steps may be indicated and the result found by cancellation,

$$.800 \times \frac{1432}{\frac{6}{3}}$$

The advantages and disadvantages are about the same as for the thirty-six-per-cent method, except that six per cent is a rate commonly used, and the result obtained is therefore often the final result.

This method may also be modified to apply to short periods of time. Thus if in discovering our rule we use six days instead of N days, we obtain

$$\frac{\$ \times P}{\$ \times 1000} = \frac{P}{1000}.$$

That is, to find the interest for 6 days at 6%, move the decimal point 3 places to the left in the principal. Illustration: Find the interest on \$324.60 for 20 days at 6%.

.3246	6 da.
.6492	12 da.
.1082	2 da.
<u>1.0820</u>	

Computing as shown at the right, we obtain \$1.082 as our result. The method used in both forms is thus applicable to long or to short periods of time.

Interest tables. — To compute interest from a table requires accuracy in reading the table, accuracy in copying, and accuracy in adding. Tables are made out in various forms and for different rates. A convenient form is illustrated on page 447.

In using the table we break the principal up into a number of hundreds plus a number of tens, etc. Thus, \$354.80 = \$300 + \$50 + \$4 + \$.80. We would then find the interest on \$300 by finding it on \$3 from the column headed \$3 and multiply the result by 100; then we would find the interest for the same time on \$5 and multiply this result by 10, and so on.

Applying this method to our problem, we proceed as follows:

P, \$800	2 yr.	.96
T, 3 yr. 11 mo. 22 da.	1 yr.	.48
R, 8%	11 mo.	.44
I, ?	20 da.	.02667
	2 da.	.00267
I, \$254.58 Ans.		1.90.934 @ 6%
		63.644 @ 2%
		<u>254.577 @ 8%</u>

INTEREST TABLE @ 6%

TIME	\$1	\$2	\$3	\$4	\$5	\$6	\$7	\$8	\$9	\$10	TIME
1 da.	.00017	.00033	.0005	.00067	.00083	.001	.00117	.00133	.0015	.00167	1 da.
2 da.	.00033	.00067	.001	.00133	.00167	.002	.00233	.00267	.003	.00333	2 da.
3 da.	.0005	.001	.0015	.002	.0025	.003	.0035	.004	.0045	.005	3 da.
4 da.	.00067	.00133	.002	.00267	.00333	.004	.00467	.00533	.006	.00667	4 da.
5 da.	.00083	.00167	.0025	.00333	.00417	.005	.00583	.00667	.0075	.00833	5 da.
6 da.	.001	.002	.003	.004	.005	.006	.007	.008	.009	.01	6 da.
7 da.	.00117	.00233	.0035	.00467	.00583	.007	.00817	.00933	.0105	.01167	7 da.
8 da.	.00133	.00267	.004	.00533	.00667	.008	.00933	.01067	.012	.0133	8 da.
9 da.	.0015	.003	.0045	.00667	.0075	.009	.0105	.012	.0135	.015	9 da.
10 da.	.00167	.00333	.005	.00667	.00833	.01	.01167	.01333	.015	.01667	10 da.
20 da.	.00333	.00667	.01	.01333	.01667	.02	.02333	.02667	.03	.03333	20 da.
1 mo.	.005	.01	.015	.02	.025	.03	.035	.04	.045	.05	1 mo.
2 mo.	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	2 mo.
3 mo.	.015	.03	.045	.06	.075	.09	.105	.12	.135	.15	3 mo.
4 mo.	.02	.04	.06	.08	.10	.12	.14	.16	.18	.20	4 mo.
5 mo.	.025	.05	.075	.10	.125	.15	.175	.20	.225	.25	5 mo.
6 mo.	.03	.06	.09	.12	.15	.18	.21	.24	.27	.30	6 mo.
7 mo.	.035	.07	.105	.14	.175	.21	.245	.28	.315	.35	7 mo.
8 mo.	.04	.08	.12	.16	.20	.24	.28	.32	.36	.40	8 mo.
9 mo.	.045	.09	.135	.18	.225	.27	.315	.36	.405	.45	9 mo.
10 mo.	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	10 mo.
11 mo.	.055	.11	.165	.22	.275	.33	.385	.44	.495	.55	11 mo.
1 yr.	.06	.12	.18	.24	.30	.36	.42	.48	.54	.60	1 yr.
2 yr.	.12	.24	.36	.48	.60	.72	.84	.96	1.08	1.20	2 yr.

We find from the column headed \$8 the interest for such periods in years, months, and days as when added will give the total time. The result is the interest on \$8. This must be multiplied by 100 to give us the interest on \$800 for the given time at 6%.

In banks and business houses where there is much computing of interest, interest tables have an obvious advantage. Elsewhere it is probably simpler to do without tables. Children in the higher grades are interested in the table method, and this interest constitutes the value of the topic.

The bankers' or sixty-day method. — This is a method for short terms only, and as the name indicates is of value

in dealing with bank loans, which are customarily for short periods of time. This method depends upon a rule which may be developed by the cancellation method. The interest on any principal at 6% for 60 days may be expressed as follows:

$$I = P \times \frac{6}{100} \times \frac{60}{360} = \frac{P}{100}$$

The rule is: To find the interest at 6% for 60 days, move the decimal point two places to the left in the principal.

To find the interest for other periods than 60 days, we break the number of days up into convenient aliquot parts of 60 or of any smaller number of days obtained in the solution. Two illustrations follow, one in which the interest is found for a number of days greater than 60, the other for a number of days less than 60. (a) Find the interest on \$192 for 112 days at 5%. (b) Find the interest on \$709 for 34 days at 10%. The solutions follow and the application of the rule is evident from the work.

(a)			(b)		
P, \$192	1.92	60	P, \$709	7.09	60
T, 112 da.	.96	30	T, 34 da.		
R, 5%	.64	20	R, 10%	3.545	30
I, ?	.064	2	I, ?	.354	3
	<u>3.584</u>	6%		<u>.118</u>	1
I, \$2.99 Ans.	.597	1%	I, \$6.70 Ans.	4.017	6%
	2.987	5%		<u>6.695</u>	10%

The bankers' method is superior to any other for short time loans. Whatever other method may be chosen as a general method, the bankers' method should by all means be taught as a special method applicable to problems

involving periods less than a year. One recent writer is such an ardent admirer of the method as to advocate its exclusive use.¹

Accurate or exact interest. — In finding the exact interest we may proceed as in the cancellation method, using 365 days to a year instead of 360. It is evident that a year's interest will be the same, no matter how many days we count to the year. That is, a year's interest on \$800 at 8% will be \$64. Hence, the modification applies only to the fractional part of the year. To illustrate, we may find the accurate interest on the problem solved in previous sections as follows :

P, \$800	330			
T, 3 yr. 11 mo. 22 da.	<u>22</u>	$\frac{8}{800} \times \frac{8}{100} \times \frac{352}{365}$		352
R, 8%	352			<u>64</u>
I, ?				1408
I for 3 yr., \$192		61.720		<u>2112</u>
A. I for 11 mo. 22 da., \$61.72		365)22528.000		22528
A. I, \$253.72 <i>Ans.</i>		<u>2190</u>		
		628		
		<u>365</u>		
		2630		
		<u>2555</u>		
		750		
		<u>730</u>		

Counting time in interest computations. — There are certain technicalities concerning the finding of time periods in interest computations that should be understood by a teacher. In general :

1. In *long terms*, the time is found by subtracting the date of the loan from the date of maturity as compound

¹ LENNES, N. J. — *The Teaching of Arithmetic*, p. 353. The Macmillan Co.

numbers are subtracted. Thus: A loan was made Oct. 6, 1921 and was paid May 1, 1924. What was the time?

$$\begin{array}{r}
 1924 \quad 5 \quad 1 \\
 1921 \quad 10 \quad 6 \\
 \hline
 2 \quad 6 \quad 25
 \end{array}$$

The result is found to be 2 yr. 6 mo. 25 da.

2. One month after a date in any month is the same date in the month following, two months after a date is the same date in the second month following, etc. Thus, a three-months loan procured on Oct. 14 is due Jan. 14 of the following year.

3. When a *short loan* is made for a number of months, it is customary in some localities to count the exact number of days from the date of borrowing to the date of maturity.

Thus, to find the time for a loan on Jan. 11, maturing on April 11, 1924, we count the exact number of days from Jan. 11 (not including the 11th) to April 11.

January,	20
February,	29 (leap year)
March,	31
April,	11
	91

The time is found to be 91 days.

If the number of days is specified for a loan, the date of maturity is found by counting the exact number of days. This may conveniently be done as follows:

Find the date of maturity for a sixty-day note given July 12.

We add 60 days to July 12, making 72 days in July. As this is an impossible date, we find it is the equivalent of 41 days in Aug. (subtracting 31, the number of days in July from 72); Aug. 41 is equivalent to Sept. 10.

The date of maturity is Sept. 10.

A second method of finding the date of maturity is superior to this if the Austrian method of subtraction is known: There remain in July 19 da.;

July 19	in August there are 31 da.	It is easy to see that the
Aug. 31	date of maturity will fall in September.	Write Sep-
Sept. (10)	tember and then draw a line and beneath it write 60	
60	as the sum. Now find what number added to 19 and	

31 will give 60. It is another illustration of finding a *missing addend* (p. 57).

The indirect cases. — In the statement: *The interest of \$600 for 3 years at 4% is \$72* there are four terms — the principal, the rate, time, and interest — any one of which may be found if the other three are given. Hence we have the following types or cases:

1. What is the interest on \$600 for 3 yr. at 4%? (\$72 missing)
2. At what rate will \$600 gain \$72 in 3 yr.? (4% missing)
3. In what time will \$600 gain \$72 at 4%? (3 yr. missing)
4. What principal will gain \$72 in 3 yr. at 4%? (\$600 missing)

In a similar way four other cases arise from the statement: *The amount of \$600 for 3 years at 4% is \$672.*

5. What is the amount of \$600 for 3 yr. at 4%? (\$672 missing)
6. At what rate will \$600 amount to \$672 in 3 yr.? (4% missing)
7. In what time will \$600 amount to \$672 at 4%? (3 yr. missing)
8. What principal will amount to \$672 in 3 yr. at 4%? (\$600 missing)

Types 1 and 5 above are called the *direct cases*, in which cases the interest must be found. Methods have already

been discussed for finding the interest. The other cases are called the *indirect cases*.

Solutions of the indirect cases. — A question illustrating any one of the indirect cases is a complex problem and may be solved by separating it into its component simple problems. In doing so, it is helpful to proceed thus: *If the time is wanting, find the interest for 1 year; if the rate is wanting, find the interest at 1%; if the principal is wanting find the interest (or amount) of \$1.* The simple problems and answers for the questions stated above as illustrations of the indirect cases are as follows:

2. What is the interest on \$600 for 3 yr. at 1%? \$18. If the interest at 1% is \$18, at what rate will the interest be \$72? 4%.

3. What is the interest on \$600 at 4% for 1 year? \$24. If the interest for 1 year is \$24, in how many years will the interest be \$72? 3 years.

4. What is the interest on \$1 for 3 years at 4%? \$0.12. If the interest on \$1 is \$0.12, on how many dollars will the interest be \$72? \$600.

6. What is the total interest if a principal of \$600 amounts to \$672? \$72. What is the interest at 1% on \$600 for 3 years? \$18. If the interest at 1% is \$18, at what rate will an interest of \$72 be produced? 4%.

7. What is the interest when a principal of \$600 amounts to \$672? \$72. What is the interest on \$600 at 4% for 1 year? \$24. If \$24 is the interest for 1 year, in how many years will the same principal yield \$72? 3 years.

8. What is the amount of \$1 for 3 years at 4%? \$1.12. If \$1 amounts to \$1.12, how many dollars will amount to \$672? \$600.

Another procedure is to express the above steps in each solution by formula, and solve by applying the formula. Thus:

P = total $I \div I$ on unit P (\$1), or $P = \text{Amt.} \div \text{Amt. on unit } P (\$1)$

R = total $I \div I$ at unit R (1%)

T = total $I \div I$ for unit T (1 yr., or 1 mo., or 1 da.)

That is, to find any missing term when the total interest is given (or can be found), divide the total interest by the interest corresponding to unity of the missing term. When the amount is given, divide the total amount by the amount of \$1. Thus, in example 2 above: Find the interest on \$600 for 3 yr. at 1% and divide \$72 by the result. $72 \div 18 = 4$. *Ans.*, 4%.

The steps in the written solutions for the indirect cases are the same as for oral solutions. A single illustration will suffice. At what per cent will \$4500 amount to \$5796 in 6 yr. 4 mo. 24 da.?

P, \$4500	1 yr.	45	4.5
Amt., \$5796	5 yr.	225	288)1296.0
T, 6 yr. 4 mo. 24 da.	3 mo.	11.25	1152
R, ? _____	1 mo.	3.75	1440
	15 da.	1.875	1440
I at 1%, \$288.00	6 da.	.75	
Total I, \$1296	3 da.	.375	
R, $4\frac{1}{2}\%$ <i>Ans.</i>		288.000	

Simple problems: What is the interest at 1%? \$288. What is the total interest? \$1296. If the interest at 1% is \$288, at what rate will the interest be \$1296? $4\frac{1}{2}\%$.

OTHER KINDS OF INTEREST

Annual or periodic interest. — *Definition.* — We have seen that simple interest is interest which is computed upon the principal only. If interest is not paid when due, it is fair that this unpaid interest should draw interest for the term that it remains unpaid. Such additional interest may be computed upon each year's unpaid interest on the principal, or it may be computed upon *all* un-

paid interest. Thus, suppose Mr. B. borrowed \$300 for 3 years at 4% and paid no interest until the end of the third year. At the end of the first year he owed \$12 interest. It was not paid, hence it should draw interest for 2 years. The next \$12 due at the end of the second year was not paid and hence interest is due on this \$12 also. He will therefore owe, besides his simple interest, the interest on \$12 for $(2 + 1)$ years, or 3 years at 4%. This is called *periodic* or *annual* interest, which may be defined as interest on the principal and on each year's unpaid interest on the principal.

Methods of computation. — Periodic interest is seldom used. Only a few states allow it. The computation for finding it is simple, and need not be illustrated.

Compound interest. — *Definition.* — In the illustration used above, besides the interest on the \$12, there will be other unpaid interest by the end of the third year. A year's interest on the first \$12, which was due at the end of the first year, will be due at the end of the second year. If not paid, this 48¢ may also be regarded as a part of the debt and bear interest during the remaining years. This is called *compound interest*, which may thus be defined: Compound interest is interest computed on the principal and on all accumulated interests not paid when due.

Methods of computation. — Let us discover a rule for finding the compound amount of any number of dollars for any length of time at any rate. The amount of \$1 at 6% for 1 year is \$1.06. This is the sum of money which will be on interest during the second year. If during the next year \$1 amounts to \$1.06, it is evident that \$1.06 will amount to 1.06 times \$1.06, or $\$(1.06)^2$. The amount at the end of the second year, $\$(1.06)^2$, is the

sum which will be on interest during the third year for the same reason as above. If during this year \$1 amounts to \$1.06, $\$(1.06)^2$ will amount to 1.06 times $\$(1.06)^2$, or $\$(1.06)^3$. It is evident that this reasoning may be continued indefinitely; hence the amount of \$1 at 6% for any number of years is 1.06 raised to the power indicated by the number of years. The reasoning is the same for *any* rate. Expressed in formula, the amount of \$1 at $r\%$ for n years is $(1 + r)^n$. Since this is the amount of \$1, the amount of any number of dollars, as P , is P times the amount of \$1, or $P(1 + r)^n$. Thus, the amount of \$800 for 4 years at 4% is $800(1.04)^4$. The compound interest may be found by subtracting the principal from the compound amount.

The written work is the same as for any written problem. The necessary computation may be performed by actual multiplication, shortening the work if possible as explained on page 234, or by use of a compound interest table in which is listed the compound amount of \$1 for any period of time up to 20 years at any ordinary rate. The first method is illustrated below for finding the compound interest on \$800 for 4 years at 4%.

		1.0816
		<u>1.0816</u>
P, \$800	$800(1.04)^4$	64896
T, 4 yr.		10816
R, 4%		86528
C. I, ?	1.04	<u>10816</u>
	<u>1.04</u>	1.16985856
C. Amt., \$935.89	416	800
C. Int., \$135.89 Ans.	104	<u>935.88684800</u>
	<u>1.0816</u>	

Let us now consider how to find the compound amount for a number of years, months, and days. We shall take 4 years 10 months 16 days at 4%. Reasoning as above, the amount of \$1 at the end of the fourth year is $\$(1.04)^4$. This is the sum which will be on interest for the fractional part of the fifth year. \$1 during this fractional part of the year will amount to \$1 plus the interest on \$1 for 10 months 16 days at 4%, or $1.035\frac{1}{3}$. $\$(1.04)^4$ will amount to $\$[(1.04)^4 \times 1.035\frac{1}{3}]$. That is, to find the compound amount of \$1 for a given number of years, months, and days at $r\%$, multiply the amount of \$1 for the number of months and days at $r\%$ by $(1 + r)^n$ where n represents the number of years and r the rate expressed in hundredths. To find the compound amount of P dollars take P times the amount of \$1 for the given time at the given rate. Thus, to find the compound interest of \$800 for 4 yr. 10 mo. 16 da. at 4% we find the amount of \$1 for 10 months 16 days to be $1.035\frac{1}{3}$. We then express the answer for the amount of \$800 for 4 years 10 months 16 days thus: $800(1.04)^4(1.035\frac{1}{3})$, or 968.75. The compound interest is \$168.75.

NOTE. — From compound interest tables (p. 469) we may find the amount of \$1 for 4 years at 4 %, and multiply this by $1.035\frac{1}{3}$.

Compound interest is sometimes payable semi-annually or quarterly. If semi-annually, reduce the given problem to a problem in which the time is *twice* the given time, the rate *one-half* the given rate, and proceed as explained above. The reason for this is obvious — there will be twice as many interest terms, and the semi-annual rate will be one-half the annual rate. For quar-

terly payments, reduce to a problem in which the time is four times the given time, and the rate is one-fourth the given rate. Thus, to find the compound amount of \$800 for 4 years 10 months 16 days at 4% interest payable semi-annually, we should proceed as in finding the compound amount of \$800 for 9 years 9 months 2 days at 2%. The arrangement of the work and the computation for finding the result is shown below.

P, \$800	4 yr. 10 mo. 16 da.	.045
T, 4 yr. 10 mo. 16 da.	2	.000 $\frac{1}{2}$
R, 4%	9 yr. 9 mo. 2 da.	.045 $\frac{1}{2}$ at 6%
C. I. if semi-ann, ?		
	800(1.02) ⁹ (1.015 $\frac{1}{2}$)	.015 $\frac{1}{2}$ at 2%
T, 9 yr. 9 mo. 2 da.	1.19509	
R, 2%	1.015 $\frac{1}{2}$	
C. Amt., \$970.52	13267	1,21.314892
C. Int., \$170.52 Ans.	597545	8
	119509	970.5184
	119509	
	1.21314892	

Savings-bank interest. — *Definition.* — Any one wishing to put his money out at interest in a safe place where he may withdraw it at any time may deposit it in a savings bank. This is an institution chartered by the state for the purpose of receiving deposits and putting them out at interest. Deposits begin to draw interest on certain days known as *interest days* — usually the first day of each month or of each quarter. Interest payments are due at stated intervals of time. The time between one interest payment and the next is known as the *interest term*. If interest is not withdrawn when due, it is credited to the depositor and draws interest the same

as an original deposit. This interest, known as *savings-bank interest*, is sometimes regarded as a form of compound interest, but there is a slight difference, as we shall see in computing it.

Method of computation. — To secure savings-bank interest a sum of money must be on deposit for an entire interest term. Interest is computed on the *smallest* amount on deposit at any one time during the interest term, and on the *exact* number of dollars, fractional parts of a dollar being disregarded.

Illustration: In a certain savings bank paying 4% interest, the interest days were Jan. 1, Apr. 1, July 1, and Oct. 1, and the interest term 3 months. If Walter A. Brown made the following deposits and withdrawals, find his balance July 1, 1921.

WALTER A. BROWN

DATE		DEPOSITS		INTEREST		WITHDRAWALS		BALANCE	
1920									
Jan.	10	400	00					400	00
Feb.	3	200	00					600	00
May	4					200	00	400	00
July	1			4	00			404	00
Aug.	10	200	00					604	00
Sept.	10					100	00	504	00
Oct.	1			4	04			508	04
1921									
Jan.	1	300	00	5	08			813	12
Mar.	10					400	00	413	12
Apr.	1			4	13			417	25
July	1			4	17			421	42

Deposits: Jan. 10, 1920, \$400; Feb. 3, \$200; Aug. 10, \$200; Jan. 1, 1921, \$300.

Withdrawals: May 4, 1920, \$200; Sept. 10, \$100; Mar. 10, 1921, \$400.

At the first interest date April 1, no interest was due because no amount had been on deposit for an entire quarter. On July 1, interest was due on the smallest balance at any one time during that quarter, \$400. On Oct. 1, interest on \$404 was due, and on Jan. 1, 1921, interest on \$508 (disregard the fractional part of a dollar); etc.

Postal Savings Bank interest. — *Definition.* — Since 1911 the United States Government has conducted a system of savings banks known as the Postal Savings Bank system. A circular containing full information as to who may deposit, and under what regulations, may be obtained from any post office or from the Postal Department, Washington, D. C.

Deposits are acknowledged by *postal savings certificates* which are made out in the name of the depositor and serve as receipts but are not salable or transferable. No certificate is issued for an amount less than \$1 nor for fractional parts of a dollar.

Method of computation. — Interest at 2% is paid each year upon the smallest balance on deposit at any time during the year. The rules for crediting interest are the same as those of savings banks, except the interest date is the first of the month next following the month in which the certificate was purchased. No depositor is permitted to have a balance in excess of \$2500. For further regulations see a postal savings circular.

Illustration : Helen Buckman has the following account in her name :

Certificates purchased : Jan. 10, 1919, \$10 ; Feb. 3, \$15 ; Mar. 5, \$10 ; Apr. 4, \$10 ; May 6, \$15 ; June 10, \$10.

Withdrawals : Mar. 4, 1920, \$20 ; May 1, \$10. What is her credit Jan. 1, 1921, if she has not withdrawn any interest before that date ?

A year's interest is due Feb. 1, 1920, on the first deposit, March 1 on the second, April 1 on the third, May 1 on the fourth. The two withdrawals would naturally be from the last three deposits, canceling the last two certificates before their interest dates, and drawing \$5 on the certificate of April 4, 1919, on the date on which 20 cents interest accrues. No other interest would be due until Feb. 1, 1921 ; hence the amount to her credit Jan. 1, 1921, is \$40.90.

Interest on daily deposits. — Under certain conditions many commercial banks allow interest at a low rate on large daily balances. Each bank makes its own rules for such interest allowance. For example, the New Rochelle Trust Company allows interest at 2% on daily balances of \$1000 or over, except in the case of accounts which because of their unusual activity require an excessive amount of banking service.

Sinking funds. — A sum of money may be set aside annually (or at regular periods) to meet an obligation at the end of a definite number of years. This is known as a *sinking fund*, and the amount of money set aside at each period is called an *annuity*. Each annuity bears compound interest from the date it is set aside until the date when the debt falls due. Thus, a municipality may

borrow \$50,000 for ten years. The sum of money which, constantly increasing by the setting aside of the annuities, will amount with interest to \$50,000 in ten years, is called the *sinking fund*. It is the amount the town must pay annually to cancel its obligation in ten years.

Tables are in common use which list the amount of \$1 set aside annually for a given number of years at different rates. Such a table is given on page 463.

To use the table we reason as follows: If \$1 set aside annually for n years at $r\%$ will amount to a given sum (amount found from table), how many dollars must be set aside annually to amount to the debt at the end of n years? Answer: The annuity desired is the number of times the amount of \$1 is contained in the amount of the debt at the end of the time. Thus: If the borough of Leonia borrows \$30,000 to be paid in 20 years, and can realize 4% on the annuity set aside, what annuity will cancel the debt? Referring to the table we find that \$1 set aside annually for 20 years at 4% amounts to \$30.96920; hence the number of dollars in the annuity is the number of times \$30.96920 is contained in \$30,000, or 968.66. The annuity is \$968.66.

EXERCISES

1. Find by the method of aliquot parts the interest and amount of the following:

(a) \$475 for 3 yr. 7 mo. 25 da. at 6%.

(b) \$349.50 for 1 yr. 10 mo. 18 da. at $5\frac{1}{2}\%$.

(c) \$472.25 from June 17, 1923, to Oct. 6, 1925, at 5%.

2. Write several periods of time expressed in years, months, and days and separate each period into convenient aliquot parts for computing interest.

3. Solve the problems of Exercise 1 by the six-per-cent method. This method is best adapted to which of the problems?

4. Find by the thirty-six-per-cent method the interest on \$435 from June 8, 1924, to March 17, 1925.

5. Find (a) by the six-day method and (b) by the bankers' method the following:

(a) Interest at 6% on \$865 for 60 days; for 42 days; for 39 days; for 75 days.

(b) Interest on \$473.50 for 93 days at 6%; at 5%; at $4\frac{1}{2}\%$; at $3\frac{1}{2}\%$; at 7%; at $7\frac{1}{2}\%$.

6. Use interest tables on as many of the above examples as lend themselves to such treatment.

7. Find the exact interest in Exercise 4; in Exercise 5 (a).

8. Find the dates of maturity for the following notes:

(a) Note dated July 5, term 90 days.

(b) Note dated Sept. 3, term 75 days.

(c) Note dated Nov. 17, term 90 days.

(d) Note dated Dec. 4, term 45 days.

9. Find the number of days for which the following *demand* notes run:

(a) Note dated Apr. 29, paid June 15.

(b) Note dated Mar. 17, paid July 5.

(c) Note dated June 16, paid Oct. 10.

10. Using problem 1 (a) write the corresponding problems (a) of type 2, (b) of type 3, (c) of type 4. Solve these problems by analysis.

11. Using problem 1 (b) write the corresponding problems (a) of type 6, (b) of type 7, (c) of type 8. Solve these problems by formula, or equation.

12. Find by actual computation, the compound interest on the following:

(a) \$600 for 4 yr. at $4\frac{1}{2}\%$, compounded annually.

(b) \$1200 for 3 yr. at 6%, compounded semi-annually.

(c) \$400 for 2 yr. at 8%, compounded quarterly.

(d) \$800 for 3 yr. 2 mo. 10 da. at 6%, compounded annually.

(e) \$900 for 1 yr. 8 mo. 15 da. at 4%, compounded semi-annually.

13. Solve, by using the table (p. 469), the problems in Exercise 12.

14. What would an annuity of \$50 set aside each year from the date of birth be worth at twenty-one years of age, if it had been invested at 4% compound interest?

15. (a) Compare the value of the compound amount of \$400 at 4% for 4 years, compounded semi-annually, with the same amount put in savings bank at beginning of an interest term and remaining for 4 years, interest payable semi-annually at 4%. (b) What would the same amount have been worth if deposited in a postal savings bank?

16. A town votes bonds to the amount of \$80,000 for erecting new school buildings. How much must be set aside annually to pay the debt in 20 years if the money can be invested in 5% securities?

17. Find the sum credited to Mr. B. Jan. 1, 1924 if his account with a savings bank paying 4% interest, interest dates Jan. 1, Apr. 1, July 1, and Oct. 1, is as follows: Deposits Jan. 10, 1922, \$50; Apr. 1, 1922, \$100; Sept. 15, 1922, \$200; withdrawals: July 1, 1922, \$75; June 10, 1923, \$60.

ANNUITY TABLE

Showing Accumulations at Compound Interest of Annual Investments of One Dollar

YEAR	AT 4%	AT 5%	AT 6%	YEAR	AT 4%	AT 5%	AT 6%
1	1.040000	1.050000	1.060000	11	14.025805	14.917127	15.869941
2	2.121600	2.152500	2.183600	12	15.626838	16.712983	17.882138
3	3.246464	3.310125	3.374616	13	17.291911	18.598632	20.015066
4	4.416323	4.525631	4.637093	14	19.023588	20.578564	22.275970
5	5.632975	5.801931	5.975319	15	20.824531	22.657492	24.672528
6	6.898294	7.142008	7.393838	16	22.697512	24.840366	27.212880
7	8.214226	8.549109	8.897468	17	24.645413	27.132385	29.905653
8	9.582795	10.026564	10.491316	18	26.671229	29.539004	32.759992
9	11.006107	11.577893	12.180795	19	28.778079	32.065954	35.785591
10	12.486351	13.206787	13.971643	20	30.969202	34.719252	38.992727

METHODS OF TEACHING

SIMPLE INTEREST

Terms. — Pupils may become familiar with the various terms used in interest and with various business usages concerning interest transactions by an informal talk, in which it will be found that any sixth-grade class of ordinary intelligence will supply much of the information themselves. Use leading questions, as, “ I know a little 6-year-old girl who has a dime savings bank of her own ; when she gets several dimes in it, she takes it to her father’s bank ; does any one know what the bank will pay her ? ” or, “ My father had a chance to buy a fine piece of property that he wanted but he didn’t have enough money to buy it ; do any of you know where he could get the money, and what he would have to do in return ? ” The different terms may be listed at the blackboard as they are mentioned in the discussion, such as *interest*, *principal*, *amount*, *usury*, *time*, etc. Different pupils may be asked to explain these terms in their own words and then read the definitions in the textbook. Thus the pupils may compare their own descriptions with the better-expressed ones in the book. They will also be ready by that time to appreciate the somewhat condensed statements that they will find in the book.

Methods of computation. — In teaching how to compute interest some preliminary drill in mental work should be given. The *thinking-aloud drill* is most helpful here. Have the pupil call off the interest for 1 year ; then the interest for the number of months by taking the fractional part of a year’s interest ; then the result. (Most mental

computations are limited to years and months, except in bankers' method.) To find the interest on \$400 for 3 years 8 months at 3%, think: "\$12, \$36, ($\frac{2}{3}$ of \$12) \$8, \$44."

Having selected the general method that seems most suitable, the teacher should make sure that in the first lessons the child *understands* the procedure; he should then look out for details in arrangement of the work, for good forms of computation, for neatness, and most of all for habits of accuracy. Plenty of practice will be needed if any degree of proficiency is to be attained. Problems involving real situations should also be used.

After a general method has been mastered, the bankers' method should be taught as a special method for short terms, and drill given upon this method. If the aliquot-parts method has not been presented, the pupils will need special help in breaking up the number of days so as to obtain *unit* fractional parts for the computation. Written work here should also be preceded by a drill in mental computation, in which the *thinking-aloud drill* is used as follows: Find the interest on \$800 for 90 days at 6%. The pupil says: "\$8, \$4, \$12." (Interest for 60 days is \$8; for 30 days is $\frac{1}{2}$ of \$8.)

The solution of problems in the indirect cases, if taught at all, should be presented as any other complex problem. The component simple problems and answers should be stated by pupils.

It is not intended that children should be taught all the methods in interest described in this chapter; nor is it considered necessary to limit the instruction to a single method. There are two plans of presenting the work that are especially commendable.

First plan. This plan is based upon the aliquot-parts method. Take up in order the following types of problems and teach each problem as indicated :

1. Finding interest on any principal at any rate for one year — a simple problem involving multiplication.
2. Finding interest for any number of years — a complex problem involving finding the interest for one year and then multiplying by the number of years.
3. Finding interest for easy fractional parts of a year — a complex problem involving finding the interest for one year and then taking the required part of this result.
4. Finding interest by the method of aliquot parts, (*a*) for years and months, (*b*) for years, months, and days.
5. The bankers' method for short periods of time, for 60 days, for 30 days, etc., then for periods over 90 days, again using the method of aliquot parts.

This procedure makes *interest* a matter of problem solution, meets adequately all practical needs, and, because of its constant use of aliquot parts, achieves unity and permits easy mastery.

Second plan. This plan involves teaching the various methods by relating each to the *cancellation* method.

1. Apply the cancellation method in problems involving any rate and any time, the time in every case being reduced to a number of days.
2. Teach problems involving short periods of time by the bankers' method. The principle of the method would be developed by cancellation as on page 448.

Instead of, or in addition to, the bankers' method the *six-day* method or the *thirty-six-per-cent* method may be taught through cancellation. There is nothing to be gained in teaching two methods, both of which

are adapted to the same type of problem; it is wise to limit the work to two, or at most, three methods.

Making an interest table. — A good exercise for a class to work out would be the making of an interest table for any rate desired, as 4%. The blanks are ruled and the desired time periods listed in the left-hand column. Then one group is assigned the column headed \$1, another the \$2 column, etc. The answers may be checked by determining whether the items in the \$2 column are twice those in the \$1 column; whether those in the \$4 column are twice those in the \$2 column, etc.

If desired, an exercise in making a table for finding length of time when the exact number of days is to be found may be worked out. Such a table may be found in most of the business arithmetics.¹ It gives the exact number of days from a date in any month to the corresponding date in any other month.

OTHER KINDS OF INTEREST

Encouraging thrift. — Pupils should be encouraged to begin at an early age to put money into a savings bank. Many schools now have banks of their own, and the computation of savings-bank interest becomes a real project as it should. The government postal savings is more convenient for some people than savings banks, and its benefits and surprising growth should be studied carefully.

Stimulating civic and social interests. — An interesting approach to compound interest is through a study of sinking funds, which can be correlated with lessons in

¹ HART, C. and WATTS, M. J. — *Commercial and Industrial Arithmetic*, p. 240; D. Appleton and Co.

taxation and civics. Issuing of bonds by municipalities and provisions for paying for them by a definite annuity should be illustrated. After computing the annuity necessary to cancel a debt, the pupils may be asked to prove that the answer is correct. Thus, suppose a town sets aside annually \$968.66 for 20 years. Find the compound amount of the first payment put on interest at 4% for 19 years; of the second for 18 years; of the third for 17 years, and so on. Add the results. What is your answer?

Interest in the subject may be aroused, too, by puzzle questions such as the following: Suppose \$1 had been put on interest (compound) at the beginning of the Christian era, what would it have amounted to by this time at 3%? W. F. White¹ gives the answer to this question for the year 1906 as approximately \$3,000,000,000,000,000,000,000,000; at simple interest the amount would be only \$58.18.

EXERCISES

1. Work out in detail the first plan (p. 466) for teaching simple interest, giving for each step a few problems graded as to difficulty.
2. Work out in detail the second plan for teaching interest.
3. Explain as to a class each of the following: (a) the *bankers' method*, (b) the *six-day method*, (c) the *thirty-six-per-cent method*, (d) the *aliquot-parts method*.
4. Make a list of drill exercises for practice on separating the *time* into convenient parts, as in the method of aliquot parts.
5. What periods of time are of most frequent occurrence in notes? What bearing have the facts upon the selection of methods to be taught? upon the selection of problems to be taught?
6. Prepare a plan for the first lesson on savings-bank interest.
7. Prepare five exercises in computing savings-bank interest, ar-

¹ *The Scrap Book of Mathematics*, p. 47; Open Court Publishing Co.

ranging them in order of complexity. Explain the new difficulty in each exercise.

8. Collect the necessary data for a project relating to provision for the retirement of a local bond issue. Discuss how you would help the children in an eighth- or ninth-year class to get a fair understanding of the situation.

9. Plan an instruction lesson on compound interest.

10. Plan a drill lesson on the use of compound interest tables in solving problems.

COMPOUND INTEREST TABLE

Amount of \$1, at various rates, compound interest, 1 to 20 years

YEARS	2%	3%	4%	4½%	5%	6%
1	1.020000	1.030000	1.040000	1.045000	1.050000	1.060000
2	1.040400	1.060900	1.081600	1.092025	1.102500	1.123600
3	1.061208	1.092727	1.124864	1.141166	1.157625	1.191016
4	1.082432	1.125509	1.169859	1.192519	1.215506	1.262477
5	1.104081	1.159274	1.216653	1.246182	1.276282	1.338226
6	1.126162	1.194052	1.265319	1.302260	1.340096	1.418519
7	1.148686	1.229874	1.315932	1.360862	1.407100	1.503630
8	1.171659	1.266770	1.368569	1.422101	1.477455	1.593848
9	1.195093	1.304773	1.423312	1.486095	1.551328	1.689479
10	1.218994	1.343916	1.480244	1.552969	1.628895	1.790848
11	1.243374	1.384234	1.539454	1.622853	1.710339	1.898299
12	1.268242	1.425761	1.601032	1.695881	1.795856	2.012197
13	1.293607	1.468534	1.665074	1.772196	1.885649	2.132928
14	1.319479	1.512590	1.731676	1.851945	1.979932	2.260904
15	1.345868	1.557967	1.800944	1.935282	2.078928	2.396558
16	1.372786	1.604706	1.872981	2.022370	2.182875	2.540352
17	1.400241	1.652848	1.947901	2.113377	2.292018	2.692773
18	1.428246	1.702433	2.025817	2.208479	2.406619	2.854339
19	1.456811	1.753506	2.106849	2.307860	2.526950	3.025600
20	1.485947	1.806111	2.191123	2.411714	2.653298	3.207136

CHAPTER XVII

COMMERCIAL PAPER

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Needs for commercial forms. — In the complicated business transactions of the present, great risk and inconvenience would be incurred if actual cash were used in discharging all debts, or if credit transactions were based upon mere verbal promises. To avoid this a great variety of legalized forms of paper have come into use, and because of the vast system of credit that pervades the modern business world, these legalized paper forms have practically the same commercial value as paper money. Let us consider some of the forms thus used.

Promissory notes. — *Definition.* — When one person borrows money from another, the borrower secures the lender by giving him a written promise to pay the debt. This is called a *promissory note*. In this written promise it is legally necessary to state (1) the place where the note is made, (2) the date of making, (3) the sum of money borrowed, (4) the time for which the note is to run (unless it is a demand note), (5) the name of the person to whom (or to whose order) the payment is to be made, (6) the proper signature of the borrower, (7) the address at which payment is to be made, and (8) the words *value received*.

The person to whom the money is payable is called the *payee* or *drawee*; the person writing the note is called the *payer* or *maker*; the sum of money for which the note is drawn is called the *face*.

Kinds of notes. — Notes may be made payable to the individual alone, or to his *order*. The former are *non-transferable* or *non-negotiable*. A non-negotiable note can be sold if the buyer has faith that the drawer will discharge the debt, but he cannot force payment by legal procedure. Notes made payable *to order* are *transferable* or *negotiable*, that is, they may be sold to another.

Interest may or may not be mentioned in the note. This gives rise to *interest-bearing* and *non-interest-bearing* notes. If a non-interest-bearing note is not paid at maturity, interest at the legal rate may be collected from the date of maturity to the date of payment. An interest-bearing note may or may not mention the rate; if no rate is designated, the legal rate for that state may be collected. In New York this is 6%. The length of time for which a note runs may be stated, or the words "on demand" may be used instead. This gives rise to *time notes* and *demand notes*.

Form. — The following shows a common form of note. The note is negotiable and interest bearing; the *face* is \$600; *payee*, Walter Brown; *maker*, G. R. Tilford; *rate of interest*, 4%; *date of note*, April 10, 1925.

The date of maturity ¹ is July 10, 1925. The amount due at maturity is \$600 plus the interest on \$600 for 3 months at 4%.

¹ In case July 10th does not fall on a legal business day, the date of maturity is the first business day thereafter, and interest may be charged for the extra day, or days.

\$600 $\frac{00}{100}$

Syracuse, N. Y., April 10, 1925

Three months -----after date, for value received,
 I promise to pay to the order of ----- Walter Brown
 at First National Bank of Syracuse

Six Hundred ~~~~~ $\frac{00}{100}$ Dollars

with interest at four per cent per annum.

G. R. Tilford

FIGURE 94

Indorsements. — As this is a negotiable note, the holder, Walter Brown, may sell the note. At the time of sale he writes his name across the back of the note (holding the left-hand edge at the top). This is called *indorsing the note*. Several forms of indorsement are possible: (1) The holder's name only may be written; this is a *blank* indorsement. It does not show to whom the note has been sold, and anyone presenting it at maturity might claim the money. (2) The holder may write above his name "Pay to the order of —," naming the person to whom he is selling the note. This is called a *full indorsement*. It shows the name of the person to whom the note was sold and gives him the right of further negotiating the note, if he so desires, or of claiming payment at maturity, if he holds it until that time. (3) The holder may write above his name "Pay to —," naming the person to whom he is selling the note; this is called a *restrictive indorsement* as the note cannot be further transferred except under the same conditions as a non-

negotiable note. It restricts the payment to the new holder.

In any one of these forms of indorsement, should the original maker fail to pay the note at maturity, each indorser is liable to following indorsers and to the final holder for the debt. If an indorser wishes to waive such liabilities, he may write the words "Without recourse"; this is a *qualified* indorsement. Any one of the first three forms may be qualified. Illustration:

BLANK	FULL
<i>Walter Brown</i>	<i>Pay to the order of B. M. Powell Walter Brown</i>
RESTRICTIVE	QUALIFIED BLANK
<i>Pay to B. M. Powell Walter Brown</i>	<i>Without recourse Walter Brown</i>

FIGURE 95

Bank discount. — *Definitions.* — When a note is sold to an individual, no general agreement prevails as to the price; but when a note is sold to a bank, the procedure involves certain technicalities which must now be explained.

The bank charges interest on the amount of the note at maturity. If the note bears interest, the amount is the face plus the interest. This interest charged by the

bank is called *bank discount* and is paid at the time the note is sold to the bank (*i.e.*, when it is *discounted*). The time for which the bank lends the money (*i.e.*, from the date of discount to the date of maturity) is called the *term of discount*. *Bank discount* may be defined as interest paid on the amount due at maturity, for the term of discount at the time of discount. In bank discount it is customary to count the time as the exact number of days; thus, in the above note if it had been discounted the day it was made, discount would have been charged for 91 days instead of three months. The amount paid the holder is the difference between the amount due at maturity¹ and the bank discount. This amount is called the *proceeds* of the note.

Illustration: Let us suppose that Walter Brown had discounted this note at the bank on April 15, at 6%. The amount due at maturity would be \$600 plus the interest on \$600 for 90 days at 4%, or \$606. This must be paid by the maker, G. R. Tilford, to the bank at maturity. The original holder, Brown, will have to pay the bank discount on \$606 for 86 days (the length of time from April 15 to July 10) at 6%, or \$8.69. The bank will therefore pay Brown the difference between \$606 and \$8.69, or \$597.31 *proceeds*. It is to be expected that he should pay the bank interest for 86 days because he will have received it for use 86 days sooner than if he had waited for payment at maturity. It does not seem logical, however, that he should pay to the bank interest on \$606 when he receives from them only \$597.31. This

¹ In the case of a non-interest-bearing note, the amount due at maturity is, of course, the face of the note.

is contrary to the usual interest agreements that interest is payable on the amount of money loaned. When one borrows money at the bank on his own note, there are three ways of figuring the interest or discount. (1) Make the face of the note equal to the desired amount plus the interest on the desired amount. For example, I want to borrow \$600. Make a note for \$606 and receive \$600. Here, interest is paid on the amount borrowed at the time I borrow the principal. (2) Make the face of the note equal to the amount due at maturity and receive the proceeds. For example, I give a note for \$600 and receive \$594. Here the interest is paid in advance on the amount due at maturity. (3) Make the face of the note equal to the amount borrowed and pay interest at stated intervals or at maturity. For example, I borrow \$600 and give my note for \$600, paying at maturity \$606. Here interest is paid on the sum borrowed at the time of settlement as in an ordinary private loan.

The following illustrates a problem in promissory notes and bank discount and its solution in written form.

A 60-day note for \$500 was given by James Smith to Henry White April 1, 1925; it was discounted at the First National Bank April 5 at 6%. (a) Who bought the note? (b) Who received the proceeds? (c) Who paid the proceeds? (d) Who must pay at maturity? (e) Whom must he pay? (f) What is the date of maturity? (g) Indorse the note in full at the time of discount. (h) Compute the proceeds.

(a) First National Bank.

(b) Henry White, holder of the note, had it discounted, and therefore received the proceeds.

(c) The bank, which discounted it, paid the proceeds.

(d) The maker, James Smith, must pay at maturity. In case he defaults Henry White must pay.

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(e) He must pay the holder, *i.e.*, the First National Bank

(f) 60 da. after April 1 or May 31.

(g)

Pay to the order of
The First Nat'l Bank
Henry White

(h)

FIGURE 96

F, \$500		
T, 60 da.	5.00	60 da.
Date, Apr. 1, 1925		
Date Dis., Apr. 5, 1925	2.50	30 da.
Rate Dis., 6%	1.66	20 da.
Proceeds, ?	.50	6 da.
	4.66	

Term Dis., 56 da.

Dis., \$4.66

Proceeds, \$495.34 *Ans.*

Suppose the above note had read "interest at 4%," the other data remaining as in the preceding problem, find the proceeds.

F, \$500	5.00	60 da. at 6%
T, 60 da.	1.666	at 2%
Date, Apr. 1, 1925	3.334	at 4%
Date dis., Apr. 5, 1925		
Rate int., 4%	5.03 _x 344	60 da.
Rate dis., 6%		
Proceeds, ?	2.5167	30 da.
	1.6778	20 da.
	.5033	6 da.
Int., \$3.34		
Amt. at mat., \$503.34	4.6978	
Term of dis., 56 da.		
Dis., \$4.70		
Proceeds, \$498.64 <i>Ans.</i>		

Bank checks. — Instead of carrying large sums of money about his person or incurring the risk of sending money, a person may deposit his money in a bank and order the bank to pay it whenever and to whomsoever he desires. When a man wishes to become a depositor in a bank, he must be introduced or identified. The bank takes his signature and address and gives him a *bank book*, sometimes called a *pass book*, in which deposits are recorded; also a book containing blank order forms or *checks*. It receives from him the amount of his first deposit which is recorded in the pass book. This is called *opening an account* with the bank.

The depositor is now entitled to draw on the bank for such payments as he may wish to make by check, up to the limit of his deposit. Some banks require all depositors to keep on deposit at all times a stipulated minimum, such as \$100 or more, or, failing in this, to pay a small monthly service charge.

A check may be made out payable to the order of cash, or to bearer, or to a specified party. In either of the first two instances any one favorably known to the bank or introduced may present the check and receive the money. In the third case the person to whose order it is payable must indorse the check when he presents it for payment. Receiving payment for a check is called *cashing a check*. A check may be indorsed in the same way as a promissory note, by the payer writing his name across the back. The indorsement may be in blank or in full; if in full, it cannot be cashed without a second indorsement by the person to whose order it is payable. A check indorsed in blank can be cashed by any one presenting it if he is

favorably known to the bank or introduced. As in a note, the person to whom a check is made out is called the *drawee* or *payee* and the one who writes it, the *maker* or *drawer*. Sometimes for cashing a check on an out-of-town bank, a small *collection fee* is charged. Illustration :

United States Mortgage & Trust Co. 1-104 <i>125th St. Branch — 125th St. at 8th Ave.</i> <i>New York, July 28, 1924</i> No. 41		
<i>Pay to the order of</i> ----- <i>W. A. Hunt</i> -----		
<i>Fifty</i> ~~~~~ $\frac{\text{no}}{100}$ Dollars		
<i>Payable through N. Y. Clearing House</i>		
$\$50\frac{00}{100}$	(Safe Deposit Bank)	<i>Robert Hartsock</i>

<i>Pay to the order of</i> <i>G. R. Bell.</i> <i>W. A. Hunt</i> <i>G. R. Bell</i>
--

FIGURE 97

The above shows a customary form of check; it was given by Robert Hartsock, a depositor in this bank, to W. A. Hunt. Below the check are indorsements — a full indorsement by Hunt when it was cashed for him by G. R. Bell, and a blank indorsement by Bell when he had it cashed at the bank.

Bank drafts. — When money is sent to another at a distance, instead of sending his personal check the sender may obtain an order from one bank upon another; this is called a *bank draft*. It is safer than a personal check as it is signed by the cashier of the bank and has therefore better security than a personal check. Thus, instead of sending his personal check, Robert Hartsock might have purchased the following draft at a bank.

$\$ 50 \frac{00}{100}$	<i>New York, N. Y., July 28, 1924</i> A. S. Mortgage & Trust Co. <i>125th St. and 8th Ave.</i> <i>Pay to the order of</i> <i>W. A. Hunt</i> <i>Fifty</i> ~~~~~ $\frac{00}{100}$ <i>Dollars</i> <i>To First National Bank</i> <i>Chicago, Ill.</i> <i>John Doe</i> <i>Cashier</i>
------------------------	---

FIGURE 98

The transaction is as follows: Robert Hartsock purchased from a bank the above draft which he sent to W. A. Hunt of Chicago. Hunt may cash the draft at any bank in Chicago, or transfer it by indorsement as in the case of a personal check. Instead of making W. A. Hunt the payee, Hartsock might have had the draft made payable to himself, and by a full indorsement made it payable to Hunt; this shows Hunt by whom the money has been sent. Banks usually charge a small premium for drafts,

such as $\frac{1}{10}\%$ of the face of the draft. If $\frac{1}{10}\%$ premium had been charged, the price of the above draft would have been \$50.05.

Bank drafts are much less used than formerly. Their chief use at the present time is by mail-order houses, many of whom demand remittance by bank draft or money order instead of by personal check.

Checks and bank drafts are passed from one bank to another until they are returned to the bank which issued them. In large cities banks settle their accounts with each other through the agency of a central system known as the *clearing house*.¹

Commercial drafts. — One method of collecting money is to send a written order to the debtor requesting him to pay the money to a designated bank, or to a third party, or to the creditor himself. Such an order is called a *commercial draft*. The draft may order the money paid at once — a *sight draft* — or paid a certain number of days after date — a *time draft*. When a time draft is presented to the debtor, he writes across the face of the draft "Accepted," the date of acceptance, and the place of payment, and returns it to the payee. When payment is due, the payee collects the money. When commercial drafts are made payable to a bank, the bank is responsible for collecting the money and charges a fee for collection.

The person who writes the draft is the *drawer*, the party to whom the money is to be paid is the *payee*, and the one upon whom the order is drawn is the *drawee*. Commercial drafts are subject to rates of exchange as are bank drafts.

¹ A good discussion of the clearing house system is given in Moore and Miner, *Practical Business Arithmetic*, p. 358.

Time drafts may be discounted the same as promissory notes. Illustration: Suppose E. W. Blakeman of Chicago owes E. L. Mendenhall of Milwaukee \$200; the latter may send the following draft to the First National Bank.

\$200 $\frac{00}{100}$	Milwaukee, Wis., July 29, 1925
Thirty days after sight.....	
Pay to the order of	The First National Bank of Milwaukee
Two hundred	$\frac{no}{100}$ Dollars
Value received and charge to the account of	
E. W. Blakeman	E. L. Mendenhall
1226 Michigan Ave.	

FIGURE 99

The First National Bank of Milwaukee sends the draft to a Chicago bank; the Chicago bank presents it to E. W. Blakeman, who writes across the face "Accepted," the date of acceptance, and the place of payment, and returns it to the bank; the Chicago Bank at the end of 30 days collects the money and sends it to the Milwaukee bank, which, after deducting collection charges, credits the money to E. L. Mendenhall. The *drawee* is E. W. Blakeman; the *drawer* is E. L. Mendenhall; the *payee* is the First National Bank of Milwaukee.

If E. L. Mendenhall had owed M. W. Ehnes \$200, he might have made Ehnes the payee, in which event his debt to Ehnes would have been cancelled when E. W. Blakeman paid the money.

This draft, when accepted by Blakeman, possesses all the properties of a promissory note. It is virtually his promise to pay the face of the draft in 30 days. Hence a time draft may be discounted just as is a promissory note. Thus, if the above draft had been made payable to M. W. Ehnes and he had wished to discount it at once, he would have had to pay the bank the discount on \$200 for 30 days at whatever rate the bank was charging. If it had been 6%, Ehnes would have received as proceeds on the draft the difference between \$200 and \$1 (the interest on \$200 for 30 days at 6%), or \$199.

Prior to the establishment of the Federal Reserve System the price of domestic exchange varied in response to changing financial conditions. To avoid shipping gold from one place to another to meet payments of drafts made on one city by another, exchange on the debtor city was offered at a discount; *i.e.*, a certain per cent below par; at the same time exchange on the other city was at a premium. Now the Federal Reserve System takes care of such matters and so equalizes the distribution of money that the price of domestic exchange no longer fluctuates.

Trade acceptances. — This commercial form (Fig. 100) has come into vogue during the past six or eight years and is taking the place of the commercial draft.

Suppose A sells B a bill of goods. He may fill out a *trade acceptance* and mail it to B. B fills out the lower left-hand corner and returns the acceptance to A. The paper is now the equivalent of a promissory note which A can discount at his bank.

Money orders. — Money may also be sent from one place to another through post offices, express companies,

TRADE ACCEPTANCE

No. 16 (CITY OF DRAWER) 192 (DATE)

To (Mr. B's name here) (NAME OF DRAWEE) (ADDRESS OF DRAWEE)

On Pay to the order of (Mr. A's, name, or A's bank) (NAME OF PAYEE)

Dollars, (\$)

The obligation of the acceptor hereof arises out of the purchase of goods from the drawer. The drawee may accept this bill payable at any bank, banker or trust company in the United States which he may designate.

Accepted at on 192 (CITY) (DATE)

Payable at (NAME OF BANK) (Mr. A's name here) (SIGNATURE OF DRAWER)

Location (Mr. B's name here) (SIGNATURE OF ACCEPTOR)

By

FIGURE 100

and telegraph companies. A person may purchase at one post office an order on another post office, to pay any sum of money (up to \$100) to a designated payee; this is called a *postal money order*. The payee cashes the money order at the post office upon which the order is drawn, or at a bank.

If the money is sent through an express company, the order is called an *express money order*, if through a telegraph company, a *telegraph money order*. The transaction in sending an express money order is similar to the sending of a postal money order, but in the telegraph money order the order is sent by telegram, the charge is greater, and larger sums may be sent in a single order. Money orders may be used in both domestic and foreign exchange.

Bills of exchange. — A draft by a person or bank in one country upon a person or bank in another is called a *bill of exchange* or *foreign bill of exchange*. If the draft is by one bank upon another, it is called a *bankers' bill of exchange*, if by one merchant upon another, a *commercial bill of exchange*. If the latter is accompanied by a *bill of lading* (the transportation company's receipt for the goods and agreement to transport and deliver same) and possibly including an insurance certificate, it is called a *documentary bill*; if no other papers accompany it, it is simply called a *commercial bill of exchange*.

Bills of exchange are subject to fluctuating rates of exchange. If one country is heavily in debt to another, she will wish to avoid the expense of shipping gold to meet the payment of drafts made upon her. Many other influences also cause fluctuations. Bills of exchange on the debtor country will therefore be sold at a discount in the

creditor country, but bills of exchange upon the creditor country will be at a premium in the debtor country.

In quoting foreign rates of exchange the monetary system of the country must be understood. These rates are quoted in the daily newspapers of large cities. A rate on Great Britain of 4.84 means that the cost of £1 is \$4.84; a rate on France of 5.1 means that the cost of 1 franc is 5.1 cents. Since the war there have been such heavy shipments of gold to this country that the gold *par* basis has largely disappeared.

Travelers' checks. — A traveler in foreign countries will wish to have available a supply of cash, without carrying large sums of money with him. One way of doing this is by means of *travelers' checks*. These are checks issued by a bank or company in denominations of \$10, \$20, \$50, \$100. Each check must be signed when purchased and signed again when cashed in the presence of the official who is to make payment. The equivalent value of the check in money of the principal foreign countries is payable at the current buying rate for bankers' checks on New York. A bank usually charges $\frac{1}{2}\%$ of face value for issuing these checks; express companies, $\frac{3}{4}\%$ of the face value. American Express Company checks are on sale at banks and are by far the most commonly used in foreign exchange.¹

Letters of credit. — A *letter of credit* is issued by a bank, and authorizes foreign banks with whom it has credit to pay sums up to a specified limit, to the named bearer or payee; *i.e.*, the person to whom the letter is issued.

¹ For an extensive quotation of foreign coins see Hart and Watts, *Commercial and Industrial Arithmetic*, pp. 365 to 370. The exchange rates there quoted are the gold standards.

The payee signs his name at the time of purchasing the letter, and is required to subscribe his name as a means of identification when he presents the letter for payment. The list of banks is printed on the letter, and the bank issuing the letter sends to these banks a description of the payee to assist in identification. A charge of about 1% of the face value is made for a letter of credit.

Domestic travelers' checks are also sold by banks; they are received as cash anywhere in the United States, upon signature by the purchaser.

EXERCISES

1. On July 10, 1922, P. W. Punnett borrowed \$600 of H. M. Evans and gave his note for this amount, payable in two years with interest at 5%. Write the note.

2. At the end of the first year P. W. Punnett sent his check to pay the interest due. Write the check, supplying proper details.

3. On Mar. 15, 1924, H. M. Evans sells the note to E. Maritt, who ten days later discounts it at the First National Bank of Hackensack. Indorse the note to show the two transfers, using a different kind of indorsement for each. Explain each indorsement.

4. If the bank charged E. Maritt 6% for discounting the note, what proceeds did he receive?

5. Considering that the above transactions have all taken place, to whom must the payment be made at maturity? By whom? Suppose that he sends this money in the form of a bank draft payable to his own order. If he purchased this draft from the Corn Exchange Bank of New York, paying $\frac{1}{10}\%$ premium, what did the draft cost him? Indorse it as he should before mailing it, using a full indorsement.

6. E. Blaker on Jan. 1, 1925, drew a draft on W. A. Hunt in favor of C. Strong, payable 30 days after sight, for \$450. The draft was accepted Jan. 3. Write the draft and explain the transaction.

7. If C. Strong had the above draft discounted at a bank on Jan. 10, what proceeds did he receive if the bank charged 6% interest?

8. Obtain from the post office and an express office prices for sending express and post office money orders, and find the cost of \$125 (a) by postal money order, (b) by express money order, from New York to Kansas City.

9. M. Latham purchases a bill of exchange in London for £25 10s. at current rate of exchange. Find the quotations in a recent newspaper and determine the cost of a demand draft at current rates of exchange.

METHODS OF TEACHING

An excellent opportunity is afforded for the socialized recitation and for real situation problems in familiarizing pupils with the different forms of commercial paper. A few suggestions follow.

Playing bank. — (1) *Notes and checks.* — The class may be divided into four groups — bankers, borrowers, lenders, and depositors. The bankers must prepare for themselves the following blank forms: checks, notes, deposit slips. Toy money is also needed. Different members of the group may act as cashiers in turn. A borrower may approach a lender and ask for a loan. He makes out his promissory note, receives from the lender a check (which the lender may procure in blank form from the banker group) for the sum borrowed, and the lender holds the note. The lender then decides to discount his note at the bank. While the banker is computing the discount on the note, the other members of the class should be doing the same thing; and if the banker makes a mistake, he surrenders his place to another member of the banker group.

A depositor makes out a check and gets it cashed at the bank. Another depositor gets a check cashed by an

individual who in turn indorses it and cashes it at the bank. At the end of the exercise, the members should check up on the transactions of the day, clear up any difficulties, and explain the transactions.

Notes may be sold to individuals instead of to the bank, if desired. The settlement at maturity for all notes given should be made before the exercise ends. If such an exercise is used for a development lesson, it will need to extend over several days. Each form as needed will be looked up in the text or explained by the teacher. If the exercise is used as a drill, familiarity with the forms will be expected, and the pupils asked to take care of the transactions without the help of the book. If a question is raised, the text may be referred to for settlement.

(2) *Other forms.* — Similar exercises may be worked out to teach bank drafts, travelers' checks, foreign bills of exchange, and so on.

Problem solving. — If any of the socialized recitations described above have been used, a variety of real situation problems will have been solved. A list of similar problems may be prepared by the pupils and used to supplement the textbook problems, or may be used instead of the textbook problems. They will probably be so stated as to convey the transactions involved more clearly than does the textbook — that is, to the children themselves.

A list of questions may be made out by the different members of the class, each one in turn conducting a drill or test. Thus, one pair may be assigned promissory notes. They give such questions as: (1) I have written a note here at the board. Is it negotiable? (2) Who is the payee? (3) Who is the maker? (4) What is the

date of maturity? (5) Suppose the note is to be sold, who has the right to sell it? (6) Suppose he sells it to William MacKenzie. Indorse it in full. (7) Who must pay the note at maturity? (8) Suppose the note is not transferred again, whom must he pay? (9) If he does not pay the amount due, who must pay it? The answer papers are then collected and marked by this pair, supervised by the teacher if he so desires. Another pair prepares a set of questions on checks, another on money orders, another on discounting notes, and so on.

Many teachers hesitate to spend the time on these topics which the socialized recitation, or the project method, requires. But the time spent in a perfunctory presentation is entirely wasted. If time is not available to handle these topics well, they should be omitted entirely. The teacher, whatever method he selects, should see to it that plenty of real-situation problems are given. They may be solved in the usual form (see p. 476) and the simple problems and answers stated as explanations.

EXERCISES

1. Plan a development lesson on bank checks.
2. Conduct a socialized recitation on promissory notes.
3. (a) Make a collection of blank forms including as many of the different kinds of commercial paper mentioned in this chapter as you can procure. (b) Be able to explain the uses of each.

CHAPTER XVIII

INSURANCE

TEACHER'S KNOWLEDGE

DEVELOPMENT OF SUBJECT MATTER

Definitions of terms. — Loss of property by fire, flood, storm, or theft ; loss of life, loss of health through sickness or accident ; loss of money through failures of firms or individuals to carry out contracts — all these and many other losses are so common that large companies have been formed whose business it is to indemnify for such losses. The agreement whereby one party, properly authorized by law, promises to reimburse another in case of loss or damage inflicted is called *insurance*. The party thus protected is called the *insured* ; the one who agrees to indemnify the other is called the *insurer* or the *underwriter*. For this protection the insured pays a stipulated sum of money, known as the *premium*. The written contract between the two parties stipulating the terms of the agreement is called the *policy*. The premiums which are paid for the protection are invested by the insurance company and supply it with funds for running the business and for paying the insurance claims.

Kinds of insurance. — Insurance against loss or damage to property is called *property insurance*. This includes fire insurance, lightning or tornado insurance, guaranty or

fidelity insurance — which is insurance against fraud or breach of contract — automobile insurance, marine insurance, and rain insurance — insurance against loss of money invested in an outdoor event in case of rain. Liability insurance insures against accident to another party, and is most often taken by employers and automobile owners.

Insurance against sickness or accident is called *health* and *accident insurance*. Insurance against loss of life is called *life insurance*.

Kinds of insurance are now so numerous that it would be impossible to make an exhaustive list. At the present time, almost any kind of loss may be covered by some form of insurance.

Property insurance. — In property insurance the maximum sum of money to be paid in case of damage may be stated in the policy; this is known as the *valued policy*. In case of damage to property, the amount of damage may be collected from the company, provided it does not exceed the amount stipulated in the policy. If the sum to be paid is not stated in the policy but is left to be determined after the loss has occurred, the policy is an *open policy*. In such a policy a limit is set to the liability. This is a form of policy often used for goods in storage. An agreement that covers several items of property, such as a group of buildings, is called *blanket insurance*, and the policy is known as a *blanket policy*.

In property insurance the premium paid is a specified rate on the *amount insured*, or is a specified rate per \$100. The rate varies according to the hazard. Premiums are payable in advance for the entire term of the policy, which

is usually a period of one, two, or three years. Policies are renewable at the end of the time or may be terminated by either party before the expiration of the time. Such termination is called *canceling the policy*. When a policy is canceled the insurer refunds to the insured a certain *pro rata* amount of the premium paid, which sum was stipulated in the policy at the time it was written.

Suppose Mr. Gapen owns property worth \$9000, which he insures for four-fifths its value at $\frac{1}{2}\%$. The amount of his insurance will be \$7200 and his premium will be $\frac{1}{2}\%$ of this amount; that is, his premium will be \$36. If his property is destroyed by fire and the loss is estimated at \$6000, the insurance company pays the \$6000, but if the loss had been \$8000, the company would have paid but \$7200, the amount stipulated in the policy.

Marine, tornado, and automobile insurance are all forms of property insurance, and the general procedure is the same for these forms as for fire insurance. For various other kinds of property insurance and a more detailed discussion see any good commercial arithmetic.¹

Rain insurance.²—Rain insurance has been an established business only about five years. In 1919 rate schedules and policy forms were first based upon a careful study of rainfall and temperature statistics over all the world.

Even after rain insurance was organized it was a losing business. In 1921 the premiums, about \$1,000,000, represented a 100% loss. Convinced that they did not know the essentials of writing rain insurance, the officials revised their methods.

¹ FINNEY and BROWN. — *Modern Business Arithmetic*, pp. 333–343.

² Quoted from the *New York Times*.

All sorts of activities, both indoor and out, likely to be affected by the weather are now regularly covered by rain insurance — picnics, circuses, conventions, sports, and so on. The policy, as a rule, covers the initial expense involved in preparation for an event, not property damage resulting from rain. Payment is made if a given amount of rain falls within a specified number of hours.

Baseball is in a class by itself. If the game is called off solely on account of the rain, payment is made for loss of income in accordance with what the gate receipts would probably have been, as shown by previous experience. If the game is played, despite the rain, insurance may be collected for diminished attendance. All forms of rain insurance must be taken out five days before the event.

Writing rain insurance is a complicated business, since rates must vary with seasons, months, and locality; and each hazard must be studied on its own merits.

Life insurance. — Various kinds of life insurance policies may be obtained.¹ The insured may pay a stipulated annual premium during his life, which guarantees at death that his heirs or any persons designated in the policy will receive the amount of the insurance. This is known as the *ordinary* or *whole-life policy*. The person to whom the money is to be paid is called the *beneficiary*.

By another form of policy, the insured is protected for a specified number of years only, such as five or ten, although at the expiration of the time the policy is renewable. The annual premium is smaller than in any other kind of policy; but if death occurs during the term of insurance, the beneficiary receives the full amount of

¹ GEPHART, W. F. — *Principles of Insurance*, Vol. I; The Macmillan Company.

the insurance, as in the *ordinary-life policy*. This is known as the *term policy*.

By paying a higher annual premium for a definite number of years, such as ten, fifteen, twenty, or thirty, the insured may have the protection for the rest of his life. That is, his beneficiary is paid the amount of the insurance at his death, even though death might not occur until several years after payments of premium had ceased. This is called the *limited-payment policy*.

Another very popular form of life insurance policy is that which guarantees to the insured a cash payment of a stipulated amount to be paid at the expiration of the term for which the policy is written. If death occurs during this term, his beneficiary receives the amount of the insurance. This is known as the *endowment policy*. At the expiration of the term he may choose another form of settlement instead of a cash payment, such as an annuity, or an increase in the amount of paid-up insurance for his beneficiary, for a specified term of years.

Some forms of endowment policies have the threefold protection: insurance paid to beneficiaries in case of death, annuity for the rest of life if disability occurs, or annuity for the rest of life after the expiration of the time for which the policy is written. The annual premium for this form of insurance is somewhat high.

The endowment policy really amounts to a combination of investment and protection. The returns are approximately equivalent to the amount that would accrue if the same money, beyond the payment needed for protection, were invested at compound interest at the rate allowed by savings banks.

In life insurance the premium is quoted at a certain amount per \$1000. This amount varies according to the kind of policy and the age of the insured, and is payable annually, semi-annually, or quarterly as the insured may elect. In carrying life insurance, other privileges besides indemnity or payment at maturity are allowed. The insured may borrow money on his policy; the sum to which he is entitled as a loan increases each year and is called the *loan value*. He may cancel his policy before maturity and receive an amount known as the *cash surrender value*. This increases annually, and is approximately equal to the sum of the premiums he has paid, with interest on them, less that portion of the premiums which paid for protection.

Life insurance companies print tables which list rates per \$1000 for different ages on the various kinds of policies. These tables are carefully worked out according to mathematical principles. The principles used are those underlying the theory of probability and chance, and the theory of sinking funds and annuities. The compilation of such a table depends, therefore, upon algebraic rather than upon purely arithmetical computations. It is easily seen that the greater the age, the greater the risk; therefore the premium increases as the age increases.

In some life insurance companies the profits of the company are shared by the policyholders. These are known as *mutual* life insurance companies. The shared profits are called *dividends*. Dividends may be paid in cash or may be deducted from the annual premium.

As an illustration let us suppose that Mr. Davis took out a 20-year endowment policy for \$5000 when he was

25 years of age. The annual premium rate for this policy at age 25 was \$49.15. Let us suppose that the average dividends for this term were \$4.86 per \$1000. Each year his annual premium on the \$5000, less the dividends, will be \$221.45 and during the entire 20 years he has the satisfaction of knowing that if death occurs during this period, his family will receive \$5000 in cash. If Mr. Davis lives, he himself will receive the \$5000 at the end of the 20 years.

Health and accident insurance. — Policies of this kind provide for payment to the insured of a specified amount in case of illness or accident. Doctor's bills and hospital expenses are also allowed in some policies. Specific amounts are mentioned for different injuries such as: loss of hand, loss of arm, loss of sight, loss of one leg, loss of both legs, and loss of finger. In case of death by accident the full amount named in the policy will be paid to the beneficiary. In some policies the indemnity is doubled in case injuries are sustained in certain specified ways, such as by a stroke of lightning, by tornado, or while on a public conveyance. These policies are written for short terms only — usually for a year or less — and are renewable.

Many forms of accident insurance policies may be obtained. Sometimes these policies guarantee protection for a few months only, or even a few days. *Travelers'* insurance policies may be taken out to insure protection during a definite trip, or for a longer period, if desired.

The premiums on all kinds of health and accident policies are payable in advance, usually for the period of a year, though in some cases provision is made for semi-annual or quarterly payments. The premiums vary

according to the benefits desired, the amount of insurance, the age of the insured, and the hazard of one's occupation.

As an illustration: Mrs. Amrine carries an accident and health insurance policy which she has had renewed every year for 10 years. After the tenth payment she had a serious illness lasting for a period of 48 weeks. During this period she was totally disabled for work for 10 weeks of the time, and partially disabled the rest of the time. She also had an operation which cost \$500. The annual premium cost was \$30 and the indemnity was (1) \$10 weekly for the first year and an increase of 10% for each year's renewal, until such accumulations amounted to 50% of the original indemnity for total disability and one-half this amount for partial disability, and (2) the payment of 300% of the original weekly indemnity for one week towards expenses of operation. Did it pay her to carry the insurance? The solution follows:

Prem. 1 yr., \$30	7.5
No. yrs., 10	38
Wk. Indem. (1), \$15 per wk. for 10 wks.	600
Wk. Indem. (2), \$7½ per wk. for 38 wks.	225
Hospital, \$30	285.0
Gain or Loss, ?	

Prem., \$300	150
Indem. (1), \$150	285
Indem. (2), \$285	30
Tot. Ind., \$465	465
Gain, \$165. Ans.	

HISTORICAL NOTES

Marine insurance. — The first appearance of insurance in business life may be said to be the marine loans of the

Greeks. Money was advanced on a ship or cargo to be repaid with large interest if the voyage was prosperous, but not repaid at all if the ship was lost. The direct insurance of sea risks for a premium paid independently of loans began in Belgium about 1300 A.D. During the next century standard rates became established for the usual voyages between London and European ports. In 1601 the English Parliament created a commission to decide disputes concerning marine insurance. The underwriters were private persons, acting independently.

In London the underwriters used to meet at fixed hours and negotiate with merchants and ship owners. Each underwriter would draw up the written agreement for that part of the loss which he assumed. Towards the end of the 17th century these meetings were held in Lloyd's Coffee House, and this practice gradually grew into the system of marine insurance now general. These underwriters at Lloyd's often assumed other than marine risks, though at this time there was in England no insurance against fire on land.

Fire insurance. — In 1635 and again in 1638 the citizens of London petitioned Charles I for a patent to insure houses against fire. It was approved, but forgotten until the great fire of 1666 revived interest in the subject. In 1680 a private fire office was opened to insure houses against fire. This insurance met a general want; the security was found to be perfect and the promise of profit great.

Mutual insurance. — The next important step was the organization of mutual insurance associations. In 1684 the Friendly Society was organized. Payments

were computed on the assumption that one house in 200 is burned every 15 years. In 1704 societies began to insure personal property.

Life insurance. — The earliest known policy of life insurance was made in the Royal Exchange, London, on the 18th day of June, 1583. The policy was drawn up for £383 6s. 8d. for a term of 12 months on the life of William Gibbons. Sixteen underwriters signed it, each for his own share, and a premium of 8% was charged. The age of the insured was not considered.

The first man to arrive at a clear understanding of the problem of life insurance was John Graunt, who published in 1661 a table "showing of 100 quick conceptions how many die within 6 years, how many the next decade, and so on for every decade up to 76." This table he prepared from the mortality registers of London. Dr. Edmund Halley, eminent mathematician and astronomer, made a more scientific study of the subject. He could find no record of ages at death in England, but obtained a register from Breslau upon which to base his study. He made an estimate of the population, as there was no census in those days, and computed the number of children in each 1000 who will die in each succeeding year after the first. He showed in parallel columns the age, the number living at that age, and the number of deaths during the year, thus forming the first mortality table. He also discovered the method of using such a table in calculating values of life contingencies, recognizing the two elements, compound interest and probability of life, which are the foundations of the theory of life contingencies.

Life insurance as a business really began with the Equitable Society of London, founded in 1762. Its premiums were computed from the Breslau table. Dr. Richard Price, a student of the new science of life contingencies, compiled (about 1780), from data taken from the parish registers of Northampton, the Northampton Table of Mortality. This made a profound impression and remained for a century the most important table of mortality. The business of life insurance began to grow, and as soon as the companies had sufficient records of their own they began to construct tables. The volumes of the Journal of the Institute of Actuaries furnish the most complete storehouse of learning on the general theory of actuarial science. The tables prepared by the Institute in 1872 still remain authoritative.

In the United States a number of professional actuaries have been associated since 1889 in the Actuarial Society of America, which has established a high standard of professional competency.

At the present time, the business of insurance has grown to such an enormous extent that indemnity for almost any conceivable loss may now be obtained, and the protection may be had for any period desired — from that of a day to that of a lifetime.

EXERCISES

1. Obtain from a local agent specific data upon some kind of insurance, and report in full.
2. Obtain from a local agent the rates for various kinds of life policies at your own age. Be prepared to make a choice, and to state the important reasons for this choice.

METHODS OF TEACHING

The life-situation problem should be realized in teaching insurance by having pupils, if possible, bring circulars which give information concerning various forms of insurance. They will be able to get from adults some data concerning property insurance. Perhaps they may bring in tables of life insurance rates. It is surprising how much information and material of this sort a class of thirty pupils can collect if the proper incentives for collecting are set up, and the children become interested in the subject.

One of the results of the study of insurance should be an understanding of insurance as an arrangement by which society protects its individual members. This idea should color all of the teaching.

Problems. — A few real situations put before the children to find out costs and benefits of different forms of insurance will give point to the work. Some companies make a specialty of children's insurance. Information and tables of insurance for minors might be used to work out the cost of insuring each pupil in the class so that at age eighteen to twenty-one the sum due would help put the girl or boy through college or a professional school. Children might become interested in trying to earn their own premium payments.

EXERCISES

1. The first lesson on *insurance* might deal with insurance in general, or with some particular kind of insurance. Take your choice and explain how you would manage such a lesson.
2. Plan a lesson on the following problem: A young man finds it

necessary to borrow money of his uncle to help himself through college. He estimates that he will require, in addition to what he can earn, about \$2000. How can this young man use life insurance to protect his uncle against loss? What is the best kind of policy for him to take out? For what amount?

CHAPTER XIX

STOCKS, BONDS, BUILDING LOANS

TEACHER'S KNOWLEDGE

DEVELOPMENT OF STOCKS

Value of a general knowledge. — A most important fact in the history of the past hundred years has been the displacement of hand labor with simple tools, by the factory system using elaborate labor-saving machinery. This revolutionary industrial development has made necessary a corresponding concentration of capital into large units. In order to build and operate an extensive railway system, or a steel business, or a power plant for the production of electric current and its transmission over long distances, a large number of persons must be persuaded to invest their money to finance the undertaking. The modern method of providing the financial support for such enterprises is the corporation.

If education is to keep pace with the movement to organize industry into larger and larger units, the schools must give increasing attention to the teaching of the principles which govern the organization and conduct of corporations. The general aims in such teaching are: (1) to secure a clearer understanding on the part of the general public of the benefits to be derived from corporate organization of industry, (2) to evolve a sane policy of

social control of corporations, (3) to develop sound judgment as to what constitutes good investment. With proper instruction in the schools the time would soon come when the demagogue could no longer make political capital out of a widespread and indiscriminate distrust of corporations, when the too frequent conspiracies between crooked business and corrupt politicians against the public interest would be less possible, and when the national waste of upwards of a billion dollars a year through foolish investments would be avoided.

A teacher should learn the general principles of modern business; he should find a little time for reading the business and financial columns in the daily papers; he should make it his business to teach pupils what they are able to understand about these matters to the end that they may be better citizens and wiser investors.

Every educated person should be able to read intelligently the financial pages of the daily newspapers — not merely for the purpose of guidance in the making of personal investments, but because financial affairs are intimately connected with the social and political developments within the nation, and also constitute an important phase of international problems. Let it not be forgotten that, while the children in our schools are too young to understand more than the simplest fundamentals of corporation finance, the effectiveness of the instruction given will be directly proportional to the depth and thoroughness of the teacher's understanding of the subject. It is an obvious, but often neglected, truth that those who undertake the task of helping our children to understand and control their environment should them-

selves be well informed concerning that environment in all its more important phases. The student of education should be also a student of life.

Definitions. — A *corporation* is a legal entity organized according to the laws of a state for the purpose of engaging in some enterprise. The legal document by which the state authorizes the organization of the corporation is called a *charter*; in it are set forth the purpose for which the corporation is formed, the rights and privileges pertaining to the corporation, etc. The *capital*¹ of a corporation comprises all the property of whatever sort, which is used in carrying on its business. A *certificate of stock*, or *stock certificate*, is a document issued by the officers of the corporation to a member, setting forth the number and kind of shares owned by the member. A *share* is a convenient unit of measure for the capital stock. A stock certificate usually specifies a fixed value for a share; this is called the *par value*, or *face value*, and is very commonly \$100. There is a growing tendency, however, to issue stock specifying *no par value*.

The illustration on the next page is a specimen stock certificate. On the back of such a certificate is a blank form to be used by John Doe in case he should wish to transfer (sell) his shares.

The name of the corporation is the Scranton Street Railway Company. The capital stock of the company consists of \$250,000, the par value of each share being \$100. John Doe acquired 200 shares of this stock on March 3, 1924.

¹ There are also corporations that do not engage in business and, therefore, have no capital stock, such as cemetery associations, lodges, etc.

No. 462

200 Shares

Incorporated under the Laws of the
State of Pennsylvania

SCRANTON STREET RAILWAY COMPANY
Capital Stock, \$250,000

Scranton, Pa., *March 3, 1924.*

This certifies that..... *John Doe* is the owner
of..... *Two Hundred* shares, of One Hundred
Dollars each, of the capital stock of the Scranton Street Railway
Company.

Transferable only on the books of the Company in person or by
attorney upon the surrender of this certificate.

Henry Smith
Secretary.

John Palmer
President.

Function of stock exchanges. — If a person wishing to buy stock in a certain corporation knew of somebody who had some of the stock that he wanted to sell, the transfer could be made without the assistance of a third person; but things are not likely to happen that way. Most of the buying and selling of shares is done through stock exchanges, which are merely associations of men who act as agents for those who wish to buy or sell stock.

The diagram in Figure 101 represents in simple form what happens when shares of stock change hands.

Mr. Seller indorsed in blank and mailed to his broker (SB) a 100-share certificate of the United States Steel Corporation, accompanied by a letter asking that the stock be sold. About the same time Mr. Buyer wrote his

broker (BB) asking him to buy 100 shares of U. S. Steel. On the floor of the stock exchange, broker SB offers the stock for sale and broker BB buys it. Mr. Buyer's name is written into the blank form on the back of the certificate, already indorsed by Mr. Seller, and the certificate is sent to the office of the U. S. Steel Corporation, where

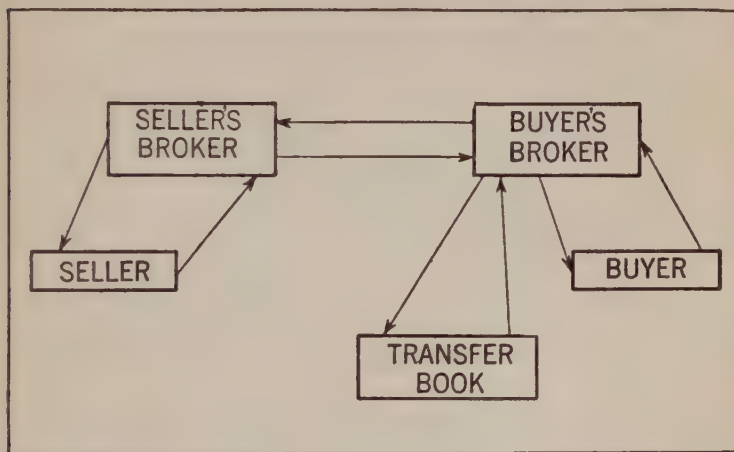


FIGURE 101

the necessary changes are made on the transfer books. The old certificate is canceled and a new certificate is issued. The lower arrows indicate the movement of the stock certificate and the upper arrows the movement of the money paid therefor. Mr. Buyer must pay to his broker the market price plus brokerage; the market price passes from BB to SB; and the market price minus SB's brokerage is remitted to Mr. Seller.

The above transaction gives only a very elementary notion of the operation of a stock exchange. If we let

Mr. Seller represent all the selling orders, Mr. Buyer all the buying orders, and SB and BB all the brokers on the exchange, we get a general idea of the complex situation which is handled through the machinery of a stock exchange.

Stock quotations. — The market value of a given stock at a given time is determined by a combination of many

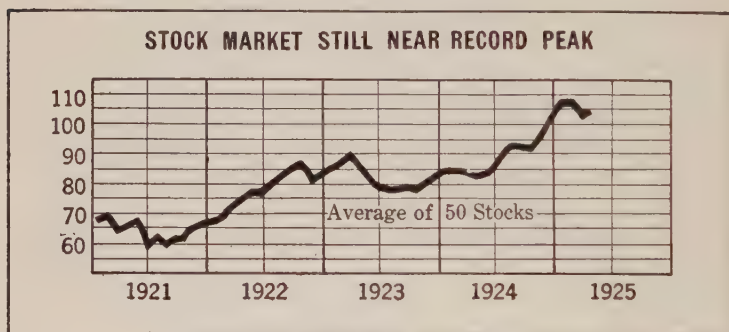


FIGURE 102

factors. Two very important factors are (1) permanence and security, and (2) earning power; these two factors are largely dependent upon the honesty and efficiency with which the business is managed. These factors are, of course, relative; a particular investment is judged to be secure as compared with other investments. It follows that a given stock may take a sharp rise or fall in market value without affecting the value of other stocks.

There are certain other factors which cause fluctuations affecting the stock market as a whole. These factors inhere in the condition of the money market, in the general tone of business and industry, in the conditions influencing international trade, etc. Figure 102 shows

NEW YORK STOCK EXCHANGE TRANSACTIONS

MONDAY, MARCH 17, 1924.

SAME PERIOD

DAY'S SALES	YEAR TO DATE	1923	1922	1921	1920
1,082,510	55,722,499	58,888,725	46,295,355	26,648,825	60,958,069

1924		SALES	STOCK AND DIVIDEND RATE	FIRST	HIGH	LOW	LAST	NET CH'GE	CLOSING	
High	Low								Bid	Ask
32 $\frac{1}{4}$	29 $\frac{5}{8}$	800	Dan Boone Wool Mill (3)	31 $\frac{1}{8}$	31 $\frac{1}{8}$	30	30	- $\frac{1}{2}$	30	30 $\frac{1}{2}$
69 $\frac{1}{2}$	46	3,300	Davison Chemical . .	51	51 $\frac{3}{4}$	50 $\frac{1}{2}$	50 $\frac{1}{2}$	- 1 $\frac{3}{4}$	50 $\frac{5}{8}$	51
112 $\frac{3}{4}$	104 $\frac{1}{2}$	300	Delaware & Hudson (9)	108 $\frac{1}{2}$	108 $\frac{1}{2}$	108 $\frac{1}{2}$	108 $\frac{1}{2}$	- $\frac{1}{4}$	107 $\frac{1}{2}$	108 $\frac{1}{2}$
119 $\frac{1}{2}$	110 $\frac{3}{4}$	1,300	Del., Lack. & West. (6)	115 $\frac{3}{4}$	116	114	114	- 1	114 $\frac{1}{4}$	115 $\frac{1}{2}$
20 $\frac{1}{4}$	17	300	Dome Mines (2) . . .	18	18	18	18 $\frac{1}{2}$	- $\frac{3}{8}$	17 $\frac{3}{8}$	18 $\frac{1}{4}$
11 $\frac{3}{4}$	11	100	Douglas-Pectin (1) . .	11 $\frac{1}{2}$	11 $\frac{1}{2}$	11 $\frac{1}{2}$	11	- $\frac{1}{4}$	11 $\frac{1}{4}$	11 $\frac{1}{2}$
141 $\frac{7}{8}$	126 $\frac{1}{4}$	7,100	Du Pont de Nemours (8)	132	132 $\frac{1}{2}$	129 $\frac{1}{2}$	129 $\frac{1}{4}$	- 3 $\frac{1}{4}$	129 $\frac{1}{2}$	129 $\frac{3}{4}$
87 $\frac{1}{2}$	78	100	Int. Harvester (5) . . .	84 $\frac{5}{8}$	84 $\frac{5}{8}$	84 $\frac{5}{8}$	84 $\frac{5}{8}$	- 1 $\frac{3}{8}$	84	85
9 $\frac{1}{4}$	6 $\frac{3}{4}$	100	Int. Mer. Marine . . .	7 $\frac{3}{4}$	7 $\frac{3}{4}$	7 $\frac{3}{4}$	7 $\frac{3}{4}$	-	7 $\frac{3}{4}$	8
34 $\frac{3}{4}$	28 $\frac{5}{8}$	2,800	Int. Mer. Marine pf..	30	30 $\frac{3}{4}$	29 $\frac{1}{4}$	29 $\frac{1}{4}$	- 1 $\frac{1}{8}$	29 $\frac{1}{4}$	30
90	83 $\frac{1}{4}$	3,600	Mack Trucks (6) . . .	86 $\frac{3}{4}$	86 $\frac{3}{4}$	85 $\frac{1}{8}$	85 $\frac{1}{4}$	- 1 $\frac{1}{2}$	85 $\frac{1}{4}$	85 $\frac{3}{4}$
90	87 $\frac{1}{4}$	100	Mack Trucks 2d pf. (7)	88 $\frac{1}{4}$	88 $\frac{1}{4}$	88 $\frac{1}{4}$	88 $\frac{1}{4}$	- $\frac{1}{4}$	88 $\frac{1}{4}$	89
109 $\frac{3}{4}$	100 $\frac{1}{4}$	1,700	Schulte Retail Stores (8)	102	102 $\frac{1}{8}$	101 $\frac{1}{8}$	101 $\frac{1}{2}$	- $\frac{7}{8}$	101 $\frac{1}{8}$	102
10 $\frac{3}{4}$	6 $\frac{1}{4}$	1,400	Seaboard Air Line . .	9 $\frac{3}{8}$	9 $\frac{3}{8}$	8 $\frac{7}{8}$	9	- $\frac{5}{8}$	8 $\frac{7}{8}$	9
22 $\frac{1}{2}$	14 $\frac{1}{2}$	2,600	Seaboard Air Line pf.	20 $\frac{3}{4}$	20 $\frac{3}{4}$	19 $\frac{1}{2}$	19 $\frac{1}{2}$	- 1 $\frac{1}{2}$	19 $\frac{1}{2}$	20
97 $\frac{1}{2}$	87	700	Sears, Roebuck & Co. .	88 $\frac{3}{4}$	89 $\frac{3}{8}$	88 $\frac{3}{8}$	88 $\frac{3}{8}$	- 1 $\frac{1}{4}$	88 $\frac{1}{4}$	88 $\frac{1}{2}$
55	38 $\frac{1}{2}$	31,100	Southern Ry. (5) . . .	53 $\frac{1}{2}$	54 $\frac{1}{8}$	53 $\frac{1}{8}$	54 $\frac{1}{8}$	+ $\frac{5}{8}$	54	54 $\frac{1}{8}$
73	66 $\frac{1}{2}$	200	Southern Ry. pf. (5) .	72 $\frac{3}{8}$	72	72 $\frac{1}{4}$	72 $\frac{1}{4}$	- $\frac{1}{2}$	71 $\frac{3}{4}$	72 $\frac{1}{4}$
18 $\frac{1}{8}$	13 $\frac{1}{4}$	300	Spicer Mfg.	13 $\frac{3}{4}$	13 $\frac{3}{4}$	13 $\frac{5}{8}$	13 $\frac{5}{8}$	- $\frac{3}{8}$	13 $\frac{3}{4}$	13 $\frac{3}{4}$
68 $\frac{1}{2}$	58 $\frac{5}{8}$	6,500	Standard Oil of Cal. (2)	61 $\frac{1}{4}$	61 $\frac{3}{4}$	60 $\frac{3}{8}$	60 $\frac{3}{8}$	- 1 $\frac{5}{8}$	60 $\frac{1}{4}$	60 $\frac{1}{2}$
42 $\frac{1}{4}$	36 $\frac{1}{2}$	5,700	Standard Oil N. J. (1)	37 $\frac{1}{2}$	37 $\frac{1}{2}$	36 $\frac{1}{2}$	36 $\frac{1}{2}$	- 1 $\frac{1}{4}$	36 $\frac{3}{8}$	36 $\frac{5}{8}$
118 $\frac{5}{8}$	115 $\frac{3}{4}$	200	Standard Oil N. J. pf. (7)	117	117	116 $\frac{3}{4}$	116 $\frac{3}{4}$	-	116 $\frac{3}{4}$	117 $\frac{1}{4}$
63 $\frac{7}{8}$	59 $\frac{1}{8}$	200	Sterling Products (5 $\frac{3}{4}$)	59 $\frac{1}{2}$	59 $\frac{1}{2}$	59 $\frac{5}{8}$	59 $\frac{5}{8}$	- $\frac{3}{4}$	58 $\frac{5}{8}$	60
100	81	20,000	Stewart-Warner Spr. (10)	85 $\frac{1}{2}$	85 $\frac{1}{2}$	81	82 $\frac{1}{8}$	- 3 $\frac{7}{8}$	82	82 $\frac{1}{2}$
84 $\frac{7}{8}$	70	1,400	Stromberg Carburetor (8)	75	75	70	70	- 6	70	72
108 $\frac{1}{4}$	98 $\frac{1}{2}$	49,500	Studebaker Co. (10) .	100 $\frac{3}{8}$	100 $\frac{1}{2}$	98 $\frac{1}{2}$	98 $\frac{5}{8}$	- 2 $\frac{1}{4}$	98 $\frac{5}{8}$	98 $\frac{3}{4}$
132 $\frac{3}{4}$	126 $\frac{5}{8}$	100	Union Pacific (10) . .	128 $\frac{1}{2}$	128 $\frac{1}{2}$	128 $\frac{1}{2}$	128 $\frac{1}{2}$	- 1	128	128 $\frac{1}{2}$
74	70 $\frac{1}{4}$	100	Union Pacific pf. (4) .	70 $\frac{3}{4}$	70 $\frac{3}{4}$	70 $\frac{3}{4}$	70 $\frac{3}{4}$	- $\frac{1}{4}$	70 $\frac{1}{2}$	71
201 $\frac{1}{4}$	182	100	United Fruit (10) . . .	194 $\frac{3}{4}$	194 $\frac{3}{4}$	194 $\frac{3}{4}$	194 $\frac{3}{4}$	+ $\frac{1}{4}$	193	196 $\frac{1}{2}$
109	98 $\frac{1}{8}$	30,000	U.S. Steel (5 $\frac{1}{2}$) . . .	102 $\frac{1}{2}$	102 $\frac{1}{2}$	100 $\frac{3}{4}$	100 $\frac{3}{4}$	- 1 $\frac{7}{8}$	100 $\frac{3}{4}$	100 $\frac{1}{8}$
120 $\frac{1}{2}$	118 $\frac{3}{4}$	300	U.S. Steel pf. (7) . . .	119	119	118 $\frac{7}{8}$	118 $\frac{7}{8}$	-	118 $\frac{1}{2}$	119

graphically the periodic character of the fluctuations in stock prices.

The slight daily fluctuations are due to a variety of causes, including what is known as *professional trading*. One cause of fluctuations is the seemingly accidental discrepancies between the volume of buying orders and the volume of selling orders for the same stock. It is plain that a great excess of selling orders will tend to depress the price, and a great excess of buying orders will tend to raise the price.

The stock quotations shown on page 509 (from the *New York Times*) illustrate the common form of newspaper reports of stock exchange transactions. The volume of sales for the day, for the year to date, and for similar periods in previous years is expressed in shares.

The meanings of the column headings are as follows: For the Dan Boone Wool Mill stock (paying at present 3 per cent annually) there were 800 shares sold; the first sale for the day was at $\$31\frac{1}{8}$ per share; the highest price paid during the day was $\$31\frac{1}{8}$, the lowest, $\$30$ a share; the last sale for the day was at $\$30$ a share; the price of the stock went down $\frac{1}{2}$ dollar per share as compared with the price for the preceding (market) day; at the close of business those who wished to buy were offering $\$30$ a share and those who were offering the stock for sale were holding out for $\$30\frac{1}{2}$ a share.

Stock brokers' commissions. — Formerly brokers charged $\frac{1}{8}\%$ commission on par value, which amounted to $12\frac{1}{2}$ cents on each $\$100$ share, irrespective of the size of the transaction. At present (1925) the rates on the New York Stock Exchange are as follows:

STOCK SELLING	PER 100 SHARES
Under \$10	\$7.50
From \$10 to, but under, \$125	15.00
From \$125 up	20.00
Minimum commission	1.00

Dividends. — It is customary for the officers and directors of a corporation to set aside at regular intervals a certain part of the net profits of the business to add to its *surplus fund* and to distribute the rest to the stockholders. When the corporation decides to distribute profits, it is said to “declare a *dividend*.” If the profits are small, it may be necessary to “*pass the dividend*,” that is, to omit declaring a dividend. A dividend may be expressed as so many dollars per share, or as a certain per cent of the capital stock. Thus, if the stock has a par value of \$100 per share a six-per-cent dividend is a dividend of six dollars per share; but if the par value were \$50, a six-per-cent dividend would be a dividend of three dollars per share. If the business requires more capital, the directors may vote to retain the profits to use as capital, and to issue to each stockholder new stock to the value of the profits withheld. Such an issue of stock is called a *stock dividend*.

Preferred Stock. — It often occurs that a corporation will find it necessary, either at the time of its organization or later, to offer special inducements to people to invest in the capital stock of the corporation. This is done by offering for sale what is called *preferred stock*. The advantage of owning preferred stock is that the owner

thereof will have the preference whenever dividends are declared.¹ The common stock can receive no dividend whatever until the preferred stock has received the dividend specifically stated on the stock certificates. Some preferred stock is *cumulative*. This means that if any dividend, or part thereof, is not paid when due, it becomes a charge payable out of subsequent profits before any dividend may be paid on the common stock.

The following was clipped from a daily newspaper :

DIVIDENDS.
248th Dividend BANK OF THE MANHATTAN COMPANY CHARTERED 1799 New York, March 13th, 1924. A quarterly dividend of THREE PER CENT (3%) and an extra dividend of ONE PER CENT (1%) from accumulated earnings has been declared on the capital stock of this company, payable on April 1st, 1924, to the stockholders of record at the close of business March 21st, 1924. The Transfer books will not close. WALTER A. RUSH, Cashier.
LEVERICH REALTY CORPORATION. 143 Montague Street, Brooklyn, New York. March 15, 1924. The Board of Directors of this Corporation has declared a semi-annual dividend of three and one-half (3½) per cent. upon the preferred stock, payable March 21st, 1924, to stockholders of record at the close of business March 15, 1924. WILLIAM L TAYLOR SECRETARY-TREASURER

¹ Preferred stock is sometimes secured as to principal and interest by a first mortgage on the property of the corporation.

The Bank of the Manhattan Company will pay dividends on April 1, 1924.

The transfer book here mentioned contains a list of all the stockholders, with their addresses, and the number of shares each stockholder owns. Whenever any shares of stock change hands the certificate held by the seller must be indorsed by him and sent to the office of the corporation. Here the old certificate is destroyed, a new certificate is issued to the buyer of the shares, and the record on the transfer book is changed.

The dividend rate of 12 per cent yearly is very attractive, but there are other things to consider when making a stock investment. Safety and permanence are of prime importance. The fact that the above company has been in existence since 1799 and that this dividend is the 248th gives evidence of safety and stability.

Safety of investments. — A study of the following real situation problems will reveal some facts concerning what constitutes a safe investment.

Problems.

1. Mr. Carl had a chance to buy stock in a company that owned a small tract of land which was believed to contain oil. The company planned to use the money which came from the sale of the stock to buy machinery and drill oil wells. Mr. Carl could have bought 100 shares at \$10 a share, but he was not a wealthy man and he decided to take no chances. A friend of his, Mr. Rich, who had plenty of money, decided to invest \$1000 in this oil proposition. How many shares did he buy?

2. It happened that most of the wells drilled the first year of operations yielded oil in large quantities. Mr. Rich might have sold his stock at \$55 a share, but refused. What would have been his profit had he sold?

3. Now that the oil had begun to flow, the company was able to sell more of its stock and at a better price; with the fresh funds they were able to drill wells in other parts of the tract. Mr. Rich bought another block (100 shares) of stock, paying \$55 a share. What did they cost him? He remarked to Mr. Carl: "They urged me to invest heavily in their company, but I told them that I preferred not to put too many eggs under the same hen." What did he mean by that?

4. This company paid dividends of \$12 a share for three years, and then the wells began to flow less freely. For the next four years the dividends were 8%, 6%, 4%, and 1%. Soon after this the flow of oil was insufficient to pay expenses and the company decided to go out of business. The land was sold very cheaply for other purposes. When the company's affairs were finally settled the stockholders got only \$2 a share. Mr. Rich had owned his first block of stock about 9 years, and his second block about 8 years. All these shares were acquired in time to participate in all the dividends mentioned above. Work out the following account and find the entire gain.

Cash investment:

100 shares at \$10, \$_____

100 shares at \$55, \$_____

Dividends received:

On one share, \$_____

On 200 shares, \$_____

Received in final settlement, \$_____

Balance gained, \$_____

5. Let us suppose that Mr. Rich had invested the money he put into oil in good mortgages paying 6%. Compute the income from the \$1000 for nine years and the income from \$5500 for eight years. How does this income compare with the oil dividends? His mortgages would still be worth their face value, as against the cash settlement received at the dissolution of the oil company. All things considered, which is the more profitable investment? Which investment is better in the long run?

6. Mr. Mitchell bought a block of shares in the oil company just

after the second dividend had been paid, when the stock was selling at \$72 a share. His money was in about six years, from the time he bought the stock to the time the company was dissolved. Using the form shown above, compute his gain or loss.

The oil company we have been discussing was a small company whose success depended upon the presence of oil in paying quantities in a rather small tract of land. An investment in such an enterprise is highly speculative. A large oil company, owning many large tracts of land scattered throughout the country, presents less risk to the investor, because low yields of oil in some tracts will be counterbalanced by high yields in other tracts. We see, then, that in the oil business (and the same thing is true of mining and other industries of the sort) organization on a very large scale gives stability to the industry and comparative safety to the investor.

DEVELOPMENT OF BONDS

Definitions. — In 1917 there was published by the Treasury Department in Washington "A Source Book" gotten out for the purpose of educating the people of the United States regarding bonds in general and to persuade them to invest in the bond issues made necessary by the World War. This pamphlet of about 55 pages is a veritable mine of information upon the subject of government finance. Copies of "A Source Book" will be found in many public libraries, and any teacher who desires to correlate history and arithmetic will do well to draw upon this source for material.

For our present purpose of definition we shall now quote from "A Source Book."

“Technically a bond is a promise to pay. A person buys a bond; that is, he lends his money to an organization and in return he receives an engraved paper receipt called a bond. A part of this bond bears the terms of the loan. It states that \$50, \$100, \$500, \$1000 or more has been borrowed from the buyer of the bond—for a stated period of years at a certain rate of interest, this interest to be paid on dates stated in the bond. The interest is usually paid twice a year. Attached to the bond are the coupons. These represent the interest. Beginning with the first date on which interest is due, these bear the date and the amount of money to be paid. If the bond was issued June 15, 1917, for \$100, at $3\frac{1}{2}$ per cent, the first coupon would state that on December 15, 1917, \$1.75 would be paid to the bearer. This coupon could be deposited or cashed at any bank.

“If a bond is registered, the owner receives a receipt for his money, the bond is held by the organization issuing it, and the interest is paid to the owner by check. Coupon bonds are payable to bearer and should be kept in a safe deposit box because they are almost the same as cash. The registering of a bond relieves the owner of the responsibility of taking care of it, though he must take care of his receipt.

“In a very general way bonds are divided into five different classes.

“I. *Government*. — A Government bond is issued by the National Government. It is secured by the wealth and taxing power of the country issuing the bond and is issued to provide funds for the maintenance of the Government, to equip the Army and Navy, build waterways,

public works of all sorts, and in times of war to provide for increased needs.

“II. *Municipal*. — A municipal bond is issued by a city, town, or village. These bonds are issued to get money for building schoolhouses, waterworks, sidewalks, fire department stations, or other public improvements voted by the municipality.

“III. *Railroad*. — Railroad bonds are subdivided into too many classes to be gone into in as brief a book as this. They are issued by a railroad company to build new lines, buy cars and engines, and to keep the roads in good condition.

“IV. *Public utility*. — Public utility bonds are issued by corporations owning and maintaining public service utilities; gas, electric light, street railways, and other such service required by the public and not owned by the city in which they are located.

“V. *Industrial*. — Industrial bonds are issued, as the name implies, by corporations manufacturing on a large scale various sorts of products. Under this class come the bonds of steel companies and other big industries.”

A bond which is secured by a mortgage on the property of the company or corporation issuing it is called a *mortgage bond*.

Investment features. — From the definitions it is seen that a bond is very like a note; hence the safety of a bond depends upon the credit of the organization — governmental unit or corporation — that issues it. In general the five kinds of bonds described in the previous section are mentioned in the order of their security; but bonds in the same general class will differ widely in this

respect. For example, certain industrial bonds rank higher than certain foreign government bonds as regards security of principal and interest. As with stocks, so with bonds, the lower the security the higher the rate of income. The quotations of May 20, 1924, which follow, illustrate the inverse relation between security of investment and rate of income.

BOND AND DATE OF MATURITY	PRICE	INCOME PER CENT OF PRICE
U. S. Liberty $3\frac{1}{2}$ s, 1932-47	99.30 ¹	3.51
U. S. Lib. 1st cv. 4s, 1932-47	100.10 ¹	3.97
U. S. Lib. 2d cv. $4\frac{1}{4}$ s, 1927-42	100.12 ¹	4.22
French Government 8s, 1945	99 $\frac{3}{4}$	
British Government $5\frac{1}{2}$ s, 1929	109	
British Government $5\frac{1}{2}$ s, 1937	101 $\frac{1}{4}$	
New York City $4\frac{1}{4}$ s, 1960	100 $\frac{1}{2}$	
City of Christiania 8s, 1945	107	
Rio de Janeiro 8s, 1946	93	
Denver & Rio Grande 4s, 1936	71	
Grand Trunk Ry. 6s, 1936	104 $\frac{1}{4}$	
Consumers' Gas of Chicago 5s, 1936	95 $\frac{3}{4}$	
Kansas City Power and Light 5s, 1952	91 $\frac{3}{4}$	
American Sugar Ref. 6s, 1937	100	
General Electric 5s, 1952	101 $\frac{1}{2}$	

Difference between interest rates and income yields. — When one buys a five-per-cent bond, he may receive more or less than five per cent interest on his investment according to whether he buys the bond at a price below or

¹ Beginning March 22, 1923, the Stock Exchange officially quoted Liberty Bonds in units and fractions, the fractions being 32ds of a unit and the quotations after the decimal point representing one or more 32ds. In the above table, therefore, the figures of highest and lowest prices during 1924 and since date of issue are reduced to the form now used in the Stock Exchange quotations.

above its face value. In this respect bonds are like stocks. But other matters must be taken into account. Note the British Government 5½s quoted above. Those which are payable in 1929 sold at 109, a \$1000 bond for \$1090, but one who buys such a bond and holds it for five years until it is paid will receive \$90 less than he paid for it;

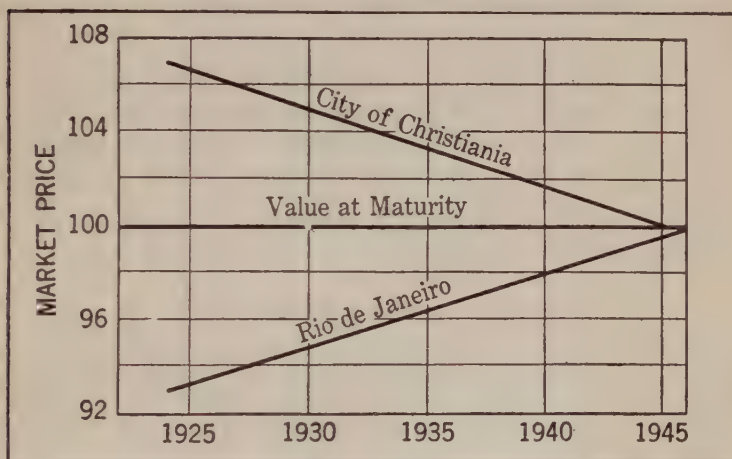


FIGURE 103

this \$90 spread over the five years virtually reduces the yearly income by \$18, or from \$55 a year to \$37 — from 5½% to 3.7% of the *face*.

There is one more point to consider in computing yields: It is the practice to take into account the fact that the present value of a sum of money to be realized at some future time is that amount which at compound interest will produce that sum. This raises slightly the result of the above calculation. People who are large dealers in bonds use printed tables in figuring yields.

The diagram (Fig. 103) represents approximately the effect upon the market price of a bond, due to the approach of the maturity date. We have selected for the illustration the bonds of the City of Christiania and those of Rio de Janeiro, because they both pay 8% interest (on face value), they mature about the same time, and the market prices quoted differ from the face value by the same amount (\$7) *but in opposite directions*.

In 21 years City of Christiania falls in market value from 107 to 100, an average of one point in three years. In 22 years the market value of Rio de Janeiro rises correspondingly. Of course, there will be constant deviations from these straight lines, owing to many influences other than the one influence considered in the diagram.

BUILDING AND LOAN ASSOCIATIONS

Aim and scope. — A building and loan association is an organization made up of two classes of people: (1) those who have borrowed money from the association to build homes and are making small monthly payments, and (2) those who are making small monthly payments as an investment. The money paid in by those who are lenders enables the association to supply the needs of the borrowers. Both classes of members are stockholders and have a voice in the conduct of the association.

This institution renders its best service in towns and cities in which the majority of the people are in moderate circumstances and live in small private houses. Such communities are very likely to be prosperous and progressive, and to attract those who have simple and wholesome standards of living.

There are at the present time (1923) in the United States 10,009 such associations having a total membership of 6,864,144 and total assets of \$3,342,530,953. The average annual payment (dues) per member is \$486.95. Thirty years ago there was a membership of only about 1,500,000 with assets of about \$500,000,000. These associations are in general subject to careful state supervision.

How a building and loan association operates. — Associations differ somewhat in organization and operation, but the following description of a certain New Jersey association will give the reader a good general idea.

The par value of the shares is two hundred dollars. As a rule, a new series of shares is offered every six months, at which time new members are received into the association. A new member, if an investor, subscribes to a certain number of shares of the stock, pays a subscription fee of ten cents a share, and agrees to make payments of one dollar a month for each share until these payments, together with the profits accruing to the shares, shall amount to the par value. The shares are then said to have *matured*. Shares mature in about eleven years.

If a new member joins with a view to borrowing money from the association, he must pay, in addition to the subscription fee and the one dollar a month to apply on the principal, interest on the money borrowed — amounting to \$1.10 per month per share; this makes the monthly payments \$2.10 a share *for the borrower*. Let us illustrate with a particular case.

Mr. X buys a building plot for \$2000, paying cash. He joins the Building and Loan Association and subscribes for 40 shares of stock.

He contracts with a builder to build a house at a cost of \$8000 — the par value of the shares. The association has a rule that the loan shall “not exceed 80 per cent of the cash value of the real estate pledged therefor.” Three appraisers, appointed by the association, inspect the plot and study carefully the building contract and the plans and specifications for the house. If they decide that the finished property will have a cash value of \$10,000 they will so report, and the association officers will vote to grant the loan.

The next step is to execute the bond and mortgage, which constitute the security for the loan, and to have Mr. X insure the house against loss by fire. The insurance policy must be deposited with the association.

Building operations can now begin; and from time to time the association will make payments to the builder in amounts approximately equal to the value of the construction work done.

EXERCISES

1. Copy and complete the table shown below, using data in report of Stock Exchange Transactions (p. 509).

NAME AND KIND OF STOCK	PRICE IN DOLLARS PER SHARE	DIVIDEND IN DOLLARS PER SHARE, YEAR	PER CENT YIELD ON INVESTMENT
Delaware & Hudson			
Dome Mines			
International Harvester			
Southern Railway			
Southern Railway pf.			
Standard Oil N. J.			
Standard Oil N. J. pf.			
Stromberg Carburetor			
Studebaker Co.			
Union Pacific			
U. S. Steel			
U. S. Steel pf.			

2. Why should Studebaker and Union Pacific stocks, both paying the same dividends, differ so much in price?

3. Explain the difference in price of Southern Railway common and preferred.

4. Select what you consider the three most conservative investments in the list tabulated above. Defend your selections.

5. How much did the buyer of the International Mercantile Marine, common, pay for his stock, including commission?

6. How much did the stock brokers make in commissions as a result of the trading in Southern Railway? In United Fruit? Remember that each share sold is also bought — two commissions.

7. Assuming that commissions average \$15 per hundred shares, compute the total commissions, on the day's transactions.

8. Compute the commissions for the sales this year to date; also the average daily commissions for this period.

9. Suppose that on March 17, 1924, you had sold 200 shares of U. S. Steel pf. and invested the proceeds in Mack Trucks 2d pf. By how much would you have increased your yearly income, assuming that both companies continue to pay dividends? Is this necessarily a good change to make?

10. On May 10, 1924, International Mercantile Marine, common, sold at \$9. If the man who bought the block of stock on March 17 sold the stock on May 10, what was his net gain, after deducting commissions and interest on his investment at 6%?

11. On May 10, Studebaker Co. stock sold at $81\frac{1}{8}$. If a man bought 500 shares on March 17 and financial difficulties forced him to sell on May 10, what was his total loss?

12. On May 10, Daniel Boone Woolen Mill sold at $24\frac{1}{4}$. On April 1 a quarterly dividend of 75 cents a share was paid to stockholders of record March 21. If I bought on March 17 and sold on May 10, what was my net gain or loss per share, counting commissions but neglecting interest?

13. Compute the per cent of income an investor would receive from an investment in each of the bonds listed on page 518, basing the per cent on the market price quoted.

14. Using the method of approximation illustrated in Figure 103 for Christiania 8s and Rio de Janeiro 8s, make the correction in annual yield from each of following: (a) Denver & Rio Grande,

(b) Grand Trunk, (c) Consumers' Gas, (d) Kansas City Power and Light. Why is this correction negligible in the case of New York City $4\frac{1}{4}$ s, and General Electric 5s?

15. In May, 1924, General Electric stock was selling around \$220 a share and these shares were paying 8% dividends. Compare these figures with the company's bonds as to price and yield. Account for the fact that investors are willing to buy General Electric bonds when their yield is less than 5%.

16. Compare the bonds of the Denver and Rio Grande and the Grand Trunk railways. Which of these yields the better rate of interest on the investment?

17. Here are verbatim copies of two bond advertisements which appeared on May 26, 1924. Mention each argument advanced in favor of the Alleghany County bonds and explain the force of it. Which of these arguments have their counterparts in the advertisement of the Minnesota bonds? What arguments in the latter are missing in the former? Which issue offers, on the whole, the best assurances of safety?

Exempt from all Federal Income Taxes

Alleghany County, N. C.

$5\frac{1}{2}$ % Road Bonds

Payable in New York City

Alleghany County, located about fifty miles northwest of Winston-Salem, covers 141,000 acres of agricultural land consisting of about 1,400 farms, practically all owned by native white farmers. Assessed Valuation, \$6,598,347. Bonded Debt, \$183,000. Population, 1920 Census, 7,403. These bonds are a direct obligation of the entire county, payable from unlimited ad valorem taxes.

Due April 1, 1952-61

YIELD 5%

EXEMPT FROM ALL FEDERAL INCOME TAXES

*Legal Investment for Savings
Banks and Trust Funds in
New York, Massachusetts,
Connecticut, Minnesota,
and other States.*

State of Minnesota

$4\frac{1}{4}$ %, $4\frac{1}{2}$ %, and $4\frac{3}{4}$ % Bonds

Due June 1, 1954

*Total Bonded Debt about 3.10%
of assessed valuation.*

$4\frac{1}{4}$ % Bonds
Price to yield 4.35%

$4\frac{1}{2}$ % Bonds
Price to yield 4.40%

$4\frac{3}{4}$ % Bonds
Price to yield 4.45%

18. How much must Mr. X (p. 521) pay into the association per month? How much per year? How much of the yearly payment is applied to principal? How much to interest?

19. Suppose Mr. X's shares mature in 11 years 6 months. How much money will he have paid in?

20. If Mr. X did not have these payments to make he could put the same amount into a savings bank at $4\frac{1}{2}\%$; hence the actual cost of his home is the sum of all the payments plus interest on these payments. Assuming that on the average the payments for a given year are made in the middle of that year, it is fair to figure 11 years' interest on the payments of the first year, 10 years' interest on those of the second year, etc. On this basis, compute the interest on all the payments. What is the total cost of Mr. X's home when it is paid for?

21. In the last question what kind of interest did you compute, simple or compound? Explain. Is this method accurate enough for the purpose in view? Give reasons.

22. A certain building and loan association has the following rule:

"Each member neglecting . . . to pay his or her dues at any meeting shall forfeit . . . the sum of ten (10) cents for each month on each share for that month. Provided, that after default in any periodical payment for three successive months, cumulative fines shall be two per centum per month on the amount in arrears."

If Mr. X misses five successive monthly payments, how much must he pay in fines?

23. Mr. Hart decided that he could save \$15 a month at least; so he subscribed for 15 shares in a building and loan association, each share \$200. If the shares matured in 11 years, how much did he actually pay in to the association? The difference between his total payments and the par value of the stock is clear gain. What is this gain?

24. We may think of Mr. Hart's investment as increasing from the amount of his first monthly payment to the amount of all his payments. Find his average investment for the 11 years. Find his average gain per year. This yearly gain is what per cent of his average investment?

NOTE.—In this problem we have used the same principle as in problem 21. The rate obtained by this method differs very little from the true rate.

REPORT OF SHARES JANUARY 31, 1924

SERIES NUMBER	SHARES ISSUED	SHARES CANCELED	SHARES IN FORCE	SHARES PLEGGED	NUMBER OF ACCOUNTS	DUES PAID EACH SHARE	PROFITS ON EACH SHARE	BOOK VALUE EACH SHARE	WITHDRAWAL VALUE EACH SHARE	SERIES NUMBER
12	164 $\frac{1}{2}$	118 $\frac{1}{2}$	46	0	10	\$132	\$59.38	\$191.38	\$188.41	12
13	73 $\frac{1}{2}$	54 $\frac{1}{2}$	19	5	6	126	54.08	180.08	177.37	13
14	153 $\frac{1}{2}$	98	55 $\frac{1}{2}$	28	11	120	49.07	169.07	166.61	14
15	174	139 $\frac{1}{2}$	34 $\frac{1}{2}$	11 $\frac{1}{2}$	5	114	44.26	158.26	156.04	15
16	153 $\frac{1}{2}$	82	71 $\frac{1}{2}$	11 $\frac{1}{2}$	9	108	39.74	147.74	145.76	16
17	114	56	58	24	9	102	35.42	137.42	133.87	17
18	199	96	103	20	19	96	31.41	127.41	124.26	18
19	245 $\frac{1}{2}$	173 $\frac{1}{2}$	72	12	13	90	27.59	117.59	112.05	19
20	249	193	56	10	11	84	24.04	108.04	103.23	20
21	100 $\frac{1}{2}$	26	74 $\frac{1}{2}$	41 $\frac{1}{2}$	10	78	20.72	98.72	92.50	21
22	188	45 $\frac{1}{2}$	142 $\frac{1}{2}$	41 $\frac{1}{2}$	24	72	17.66	89.66	84.34	22
23	33	10	23	0	3	66	14.84	80.84	74.90	23
24	236	48	188	49	26	60	12.26	72.26	67.34	24
25	400 $\frac{1}{2}$	131	269 $\frac{1}{2}$	134 $\frac{1}{2}$	25	54	9.93	63.93	58.96	25
26	1700	685	1015	176	109	48	7.85	55.85	51.92	26
27	532	170	362	117	38	42	6.01	48.01	44.40	27
28	1148 $\frac{1}{2}$	236 $\frac{1}{2}$	912	221	84	36	4.41	40.41	37.76	28
29	660 $\frac{1}{2}$	145 $\frac{1}{2}$	515	209 $\frac{1}{2}$	47	30	3.06	33.06	30.91	29
30	768 $\frac{1}{2}$	118 $\frac{1}{2}$	650	199	68	24	1.96	25.96	24.59	30
31	424 $\frac{1}{2}$	77 $\frac{1}{2}$	347	61	36	18	1.10	19.10	18.27	31
32	801	25	776	120	81	12	.50	12.50	12.13	32
33	423	0	423	140	35	6	.12	6.12	6.00	33
	8942 $\frac{1}{2}$	2729 $\frac{1}{2}$	6213	1632	679					

25. A certain building and loan association gives the above report: In its constitution appear the following rules concerning withdrawals:

"Any shareholder wishing to withdraw from this association may do so by giving a written notice five days prior to the regular meeting of the Board of Directors.

"During the first year of his or her series of stock he or she shall be entitled to receive the actual amount of dues paid in . . .

"During the second year, dues and 25 per cent of profits . . .

"During the ninth year, dues and 90 per cent of the profits . . .

"During the tenth year, and until the series closes, dues and 95 per cent of profits . . ."

Study the table on page 526 and show how the withdrawal values are computed.

From the same table work out the following questions:

26. Find the average number of shares per member. Each member has an "account."

27. By mental computation find approximately what proportion of the shares of each series are *pledged* — held by borrowers. What per cent of all the shares in force are pledged?

28. On the date of the report how many monthly payments had been made on the shares of series 22? Of series 15? Of series 12?

29. Find the per cent profit on the payments made during the first year, figuring on an interest basis.

Solution: Series 32 has been outstanding one year. \$12 has been paid on each share — \$1 has been in 12 months, \$1 has been in 11 months, \$1 for 10 months, and \$1 for 1 month; this is equivalent to an investment of \$1 for 78 months, or $6\frac{1}{2}$ years. If the profit on an investment of \$1 for $6\frac{1}{2}$ years is \$.50 the profit on \$1 for 1 year is \$.077 — or the profit on payments made during the first year is 7.7 per cent of the investment.

30. Find, as above, the per cent return on payments made during the eleventh year. Note that the profits for any one year are the difference between the total profits accruing to the share at the beginning of the year and the total profits accruing at the end of the year; the profits column is cumulative as it stands.

METHODS OF TEACHING

STOCKS

Methods and aims in general. — The method to be used in teaching the subject of stock investments is essentially the laboratory method. Every effort must be made to get into close contact with the real situation. The

textbook in arithmetic can help by giving clear definitions of the terms used and realistic descriptions of business practices followed. In the seventh or eighth year of the elementary school only the simplest elements of the subject can be taught. These are (1) the nature and function of a corporation, (2) the essential characteristics of a sound investment, (3) the solution of such problems as arise in the experience of the investor (not the speculator). Anything beyond these bare essentials must be deferred until the children are more mature. It is unfortunate that the high schools so rarely make room for the subject except in commercial courses.

Teaching procedures. — A few concrete suggestions as to teaching procedure are here in order :

(1) Some teachers find it helpful to dramatize the work. The class might, for example, be organized into a corporation, chartered by the principal, for the purpose of conducting a season of athletics or an entertainment. Such an experience would help to convey to the pupils the meaning and function of a corporation.

(2) The collection and reading of news items and advertisements will serve as a good basis for discussing the question of what constitutes a sound investment.

(3) The buying of stocks and the receiving of dividends may well be studied as follows :

Let the subject be taken up early in the term. As soon as the general ideas are sufficiently understood have each child make believe that he has \$1000 to invest in stocks. Each selects one or more stocks to buy, and writes a letter to his broker (teacher) ordering the purchase of the stock. During the term, or year, the children make record of

any dividends declared by their corporation and how much is coming to them. They sell the stock the last week of the term or year at the market price and then figure out their gains or losses. This plan provides for a rather careful study of the subject at the beginning of the term, for occasional reviews during the term, and for a good rounding off of the subject at the end. It furnishes motive for becoming familiar with the market reports in the daily papers, and for the solution of the following types of problems:

How many shares of —— can I buy for \$—— at to-day's price? What will they cost including brokerage?

When are the dividends paid, and how many may I expect to receive? I must credit the investment with each dividend as it is declared.

What are the net proceeds of the sale of my shares at the price quoted on the day of sale (agreed upon by the class in advance)?

Prepare a statement of expenditures and receipts and find the net gain or loss.

During the same period I might have put this money in the savings bank at 4 per cent. Would this have been more, or less, profitable than the stock investment? Find the difference between the two.

BONDS

Teaching procedure. — The point of departure for the study of bonds is the pupils' knowledge of interest-bearing notes. By comparing a bond with a note the pupils would discover the essential points of similarity between the two. The features of a bond not present in a note should then be studied. To vitalize the problem work the teacher can follow procedures analogous to (2) and (3) on page 528.

BUILDING AND LOAN ASSOCIATIONS

Aims. — The ends to be sought in teaching this topic are : (1) to acquaint the children with a convenient method of investing *small* savings, (2) to develop a correct sense of the far-reaching individual and social benefits to be derived from such investments, and (3) to stimulate the child to begin the practice of making investments.

Suggestions. — Where there is a local Building and Loan Association the best introduction to the subject would be the reading and discussion of the literature and reports issued by the association. Out of this discussion would grow the desire to solve certain problems of the home builder, of the investor, and of the association officers. Much of the material under *Teacher's Knowledge* is adaptable to the requirements of an eighth- or ninth-grade class, but the less important technicalities and the more difficult problems should be avoided. The teacher must remember that she is not responsible for the training of a technician, or even a teacher, but is concerned with the development of an intelligent and public-spirited citizen.

EXERCISES

1. Work out in detail a plan for the first lesson on stock investments.
2. Tell in detail how you would proceed in getting the children interested in financial news. Show concrete material. State the problems that would arise.
3. Outline a project which would give occasion for the formation of a make-believe corporation in the class, and explain how the project would be carried out.
4. Work out in detail a term project along the lines suggested in (3), page 528.

5. The following is taken from a folder compiled and distributed by the New Jersey Bankers' Association for the purpose of protecting their depositors against unwise investments. Study the introduction and the questions, and then see if you can explain the idea behind each question.

"This Questionnaire was compiled by the New Jersey Bankers' Association to protect the small investor and to emphasize the necessity of investigating thoroughly every investment proposition before placing any money therein. When you have been furnished with the information called for in this Questionnaire, then take it to any one of the banking institutions in your community and get their opinion of the stock offered. If the stock salesman refuses to fill out this Questionnaire, have nothing more to do with his proposition, as it is evident that he is fostering a fraudulent promotion."

Date.

Name of company.

Name of salesman.

Kind of stock offered.

Total issue of stock.

Amount of stock issued in return for property.

Amount of stock issued for good will.

Amount of stock issued in return for patents.

Do you take Liberty Bonds in payment?

If so, at what price?

Par value of stock.

Market price of stock.

Has the stock a ready market?

If so, where is it listed?

Is the stock accepted by banks as collateral for loans?

If so, what banks have accepted it to your knowledge?

What are the present net earnings?

Bank references.

Names and former occupations of officers.

6. Work out a plan for a first lesson on bonds.

7. Collect material from current newspapers and other sources and plan a project involving the study of bonds.

8. Under what conditions would it be wise to undertake the teaching of loan associations? Prepare an outline covering as much of the topic as you would teach, basing the work upon the literature issued by the local association. Divide this material into convenient lesson units.

CHAPTER XX

TAXATION—THE FINANCIAL SUPPORT OF GOVERNMENT

TEACHER'S KNOWLEDGE

Aims in the study of taxation. — It is estimated that the people of the United States in the year 1924 expended for the support of government—local, county, state, national—the tremendous total of \$9,800,000,000. Assuming this burden to be spread over a population of 110,000,000 it is easy for one to figure the average per person and the average per family of four or five persons. Not only are taxes high, but they are on the increase—particularly in the case of governments of cities of 30,000 population or more.

Here is how the cost payments of city governments have risen, according to census bureau figures and to estimates based on available figures:

1916.....	\$1,068,301,311
1917.....	1,108,021,565
1918.....	1,172,695,829
1919.....	1,233,111,835
¹ 1920.....	1,500,000,000
¹ 1921.....	1,800,000,000
1922.....	2,222,566,519
¹ 1923.....	2,360,000,000
¹ 1924.....	2,500,000,000

¹ Estimated.

In view of these facts it is clearly the business of every citizen, *and of every prospective citizen*, to be informed about taxation — to know (1) for what purposes his contribution is used, (2) whether the money is efficiently administered and wisely spent, (3) how his share of the total tax is collected, (4) whether there is a fair distribution of the burden so that each individual pays approximately in proportion to the benefits derived. Since these questions have pronounced mathematical aspects it should be the aim of the public school in its teaching of civic duties and responsibilities to present the *mathematics* of taxation as well as the other phases of the subject.

Definitions. — Money raised for the purpose of defraying the expenses of government is called a *tax*. Taxes are of comparatively recent origin. The medieval state depended for revenues largely upon the products of the public domain. Taxes were first developed in cities. The more highly civilized a state or government, the heavier the taxation. Everybody, at the present time, directly or indirectly pays taxes. A *direct tax* is a tax levied upon a *person*, or upon his *property* or *income*. In some states a *poll*, or *head, tax* is levied upon every person of voting age; the federal government and many of the state governments levy taxes upon the incomes of individuals and corporations — *income taxes*; states and lesser governmental units levy taxes on real estate or personal property or both; these, together with minor taxes on dogs, automobiles, etc. are all *direct taxes*.

Indirect taxes include *duties* on imported goods, *internal revenue taxes* on alcoholic liquors, tobacco, etc. It is obvious that every consumer of an article thus taxed must

pay the price increase due to such tax. Moreover, every dollar of taxation levied upon income-producing property must in the last analysis be paid by the general public that lives in rented dwellings and patronizes the business enterprises upon whose property direct taxes are levied. *There is no escape from taxation.*

Property taxes. — Let us illustrate the procedure in levying and collecting the taxes for the support of a town or city. The following are the essential steps in the process.

The budget. — According to the best modern practice a committee, or board, constituted for the purpose, prepares a detailed statement setting forth the estimated expenditures for the coming year for each of the several departments of the city government — education, police, fire, health, correction — and also for payment of principal and interest on outstanding loans (bonds, etc.). This statement is called the *budget*. The amount of the budget, less any items of income, is the amount to be raised by taxation.

Each municipality must also bear its share of the state and county tax. The sum of these items is the *total tax* levied.

Assessment of property values. — Since each property owner is to be taxed in proportion to the value of his property, it is necessary to prepare a list of property owners and to set opposite each name (of individual or corporation) a fair estimate of the value of the property. This estimate is called the *assessed value*, and the officials who prepare the list and fix these values are called *assessors*.

Determining the tax rate. — It is now necessary to compute the ratio of the total tax to the total value of all the taxable property. This ratio is called the *tax rate*; it may be expressed as a per cent, but is more commonly expressed as so many mills on a dollar or as so much per \$100 or per \$1000.

Collecting the tax. — Having determined the rate of taxation, the next step is to compute the amount of tax to be paid by each individual and corporation on the tax list and to prepare the tax bills.¹ The official in charge of the collection of taxes is usually called the *collector*.

To facilitate the computation of the tax bills it is customary to prepare a tax table like the following. (The student may complete the table.)

TAX TABLE. RATE \$2.13 ON \$100

PROPERTY	TAX	PROPERTY	TAX	PROPERTY	TAX
\$10	\$0.213	\$100	\$2.13	\$1000	\$21.30
20	.426	200	4.26		
30					
40					
50					
60					
70					
80					
90					

State and county taxes collected locally. — It is a common practice to collect all property taxes, including county

¹ In many places no tax bills are sent out. Public announcement is made that taxes are due and will be received at certain specified times. The taxpayers are required to render payment at the office of the collector.

and state taxes, at the same time and through the same official machinery. This makes for efficiency and economy. An actual instance will illustrate the operation of this scheme.

The following tax rate is quoted from a tax bill for the year 1924:

RATES PER \$100	
State Road	\$.1027
State School	.2598
Soldiers' Bonus Bond Tax	.0221
State Bridge & Tunnel Tax	.0208
State Hospital Tax	.0500
County	.9225
District Court	.0051
Local	1.7758
School	2.1212
Total	<u>\$5.2800</u>

A resident in this town had assessments as follows: land, \$2250; buildings, \$2600; personal, \$100; automobile, \$200; poll tax, \$1. The solution of the problem for finding his total tax is shown below.

Land,	\$2250		What is the total value of the
Bldg.,	2600		property? <i>Ans.</i> \$5150.
Personal,	100	5.28	If the tax rate is \$5.28 on \$100,
Auto,	200	51.5	what is the tax on 51.5 hundred
Poll,	1	2640	dollars? <i>Ans.</i> \$271.92.
Rate,	\$5.28 per \$100	528	
Tot. tax, ?		2640	What is the total tax? <i>Ans.,</i>
		<u>271.920</u>	\$272.92.
Tot. val.,	\$5150		
Prop. tax,	\$271.92		
Tot. tax,	\$272.92	<i>Ans.</i>	

Income taxes. — The first federal income tax law was passed in 1913. It imposed a tax of one per cent on the

excess of income over \$3000 for single individuals, and over \$4000 for husband and wife. An additional tax was charged on incomes in excess of \$20,000.

The income tax law has been twice revised since 1913—in 1917 and in 1924. The varying needs of government, together with the constant agitations for adjustments, will probably bring about further revisions of the law.

The revenue act of 1924, effective January 1, 1925, imposes a normal tax on individuals who are citizens or residents of the United States at the following rates: two per cent on the first \$4000 of net income in excess of the credits allowed; four per cent on the next \$4000 of such excess; and six per cent on the balance of such excess.

This act fixes the exemption of single persons at \$1000, of married persons or heads of families at \$2500, of married persons or heads of families with one dependent at \$2900, two dependents at \$3300, three dependents at \$3700. A surtax is charged on all incomes over \$10,000.

The table on the opposite page shows the amount of the tax due on various incomes for single persons and heads of families.

Additional exemptions are allowed on income taxes for such items as state and municipal taxes, dividends on stocks or the earnings of a corporation which is taxable upon its net income, necessary expenses paid out for carrying on business, and so on.

The following problem and its solution illustrate a simple case:

Compute the normal tax of a married man with no dependents, receiving a salary of \$15,000.

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The first \$5,000 of net income in all cases is deemed to be earned.
In no case may the earned net income exceed \$10,000.

Surtax on incomes in excess of \$500,000 is 40%
Compiled by Joseph J. Mitchell, Editor Income Tax Review.

Salary	\$15000		
Exemption	\$2500		
Tax	?		
		8000	
Tax on 1st \$4000	\$80	<u>2500</u>	80
Tax on 2d \$4000	\$160	10500	160
Balance	\$4500	15000	270
Tax on balance	\$270	<u>10500</u>	<u>510</u>
Total tax	\$510	Ans. 4500	

Indirect taxes, or duties.—*Definitions.*—Customs duties are levied either upon the price of the goods or upon the amount of the goods, giving rise to *ad valorem* and *specific* duties. An *ad valorem duty* is a duty levied upon the cost or value of imported goods. It is expressed as a per cent of the value of the goods as determined by government officials known as *appraisers*. *Ad valorem* duties are not computed on fractional parts of a dollar; fifty cents or more is called an additional dollar; and a fractional part less than fifty cents is disregarded.

A *specific duty* is a duty which is a specified amount per unit of measure, such as a pound, ton, yard, gallon, etc. Specific duties are not computed on fractional parts of a unit; less than half is disregarded, a half or more is called an additional unit.¹ Some imported goods are subject to both an *ad valorem* and a *specific* duty. An office established and maintained by the national government for collecting duties at ports where ships enter is called a *customhouse*. The officials who take charge of the business are the *appraisers*, the *collectors*, who look after the papers and collect the duties, and the *surveyors*

¹ Formerly an allowance was made for waste, known as *tret*; and a second allowance known as *tare* for weight of a container, such as box, bag, cask, etc.

who take charge of the vessel and cargo and deliver the merchandise to the importer.

A detailed statement of the goods imported is made out in the weights and measures, or in the currency, of the country in which the goods were bought and is called an *invoice*.

The master of the vessel must make out a memorandum, called a *manifest*, stating vessel's name, cargo, port of departure and of entry, and the names of the shippers and importers. Buildings are provided by the Government known as *bonded warehouses* for storing the goods on which duty has not been paid.

Problem solved at present rates. — A few items selected from the Fordney-McCumber law of 1924, showing the tariff rates, are quoted below :

Printing paper (lb.)	$\frac{1}{4}\text{¢} + 10\%$
Writing paper (lb.)	$3\text{¢} + 15\%$
Pencils (gross)	$45\text{¢} + 25\%$
Refill leaders (gross)	10¢
Leather-bound books	30%
Printed books (lb.)	7¢
Post cards, not including American views	30%

Suppose a stationer imports 120 lb. of writing paper invoiced at 23¢ a pound, 40 gross of pencils invoiced at \$2.40 a gross, 120 leather-bound volumes at \$1.19 each, and $42\frac{1}{4}$ lb. booklets. What is his total duty ?

Paper, 120 lb. at 23¢	
Pencils, 40 gross at \$2.40	.23
Books, 120 at \$1.19	120
Booklets, $42\frac{1}{4}$ lb.	27.60
Tot. D, _____ ?	

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Val. paper,	\$27.60	.28	24.00	In computing the <i>ad</i>
Sp. duty p.,	3.60	15	3.60	<i>valorem</i> duty the dutiable
Ad V D p.,	4.20	<u>4.20</u>	4.20	value of the paper,
Sp D pen.,	18.00	1.19	18.00	\$27.60, is counted as \$28;
Val. pen.,	96.00	120	42.90	of the books, \$142.80 is
Ad V pen.,	24.00	<u>142.80</u>	2.94	counted as \$143.
Val. books,	142.80	1.43	95.64	The specific duty on
Ad V D b.,	42.90	30		the booklets is computed
Sp D bkl.,	2.94	<u>42.90</u>		on 42 lb.
Tot. D,	\$95.64	<i>Ans.</i>		

EXERCISES

1. How do you account for the rapid rise in city taxes from 1916 to 1924? Represent graphically the facts on page 533.
2. Prepare a tax table using the rate \$5.28 per \$100, quoted in the schedule cited on page 537.
3. Express the rate for each item in the same schedule (*a*) in terms of mills on \$1, (*b*) as dollars on \$1000. Why is the rate used more convenient than the mills on a dollar rate? See page 536.
4. There were in the United States 6,787,481 people who paid federal taxes on incomes received during 1923. What per cent of the population were subject to this tax?
5. During the same year (1923) the total of the net incomes taxed was \$21,336,212,530. Find the average net income of those who paid the tax.
6. Compute the income tax of a man who earns a salary of \$13,000 a year, has a family of wife and three children, and has no unearned income.
7. Compute the income tax of a married man with one dependent, who receives a salary of \$5000 a year and gives 10% of his income to charities.

METHODS OF TEACHING

General suggestions. — Material for presenting practically everything in *taxes* as real situation problems may be obtained in abundance. Tax bills supply in-

formation for property taxes and poll taxes. The printed information on the back of a New York City tax bill affords enough problem material — in other lines as well as taxation — to serve in lieu of pages of textbook problems. A collector of internal revenue will supply information concerning income taxes. Newspapers and magazines print tariff rates on imported goods. Commercial arithmetics of recent date serve as valuable reference books.

EXERCISES

1. Explain as to a class how one who buys a coat at a city department store incurs a share in that city's tax burden.
2. Prepare a plan for a first lesson on property taxes, adapting the work to your community.
3. Prepare a plan for a first lesson on duties.
4. Outline what you might appropriately teach eighth-grade children about income taxes. Describe a first lesson on the subject.
5. Make a series of five problems on income tax, using current rates and graded in difficulty so as to be suitable to eighth-grade children. Solve these problems.
6. Prepare a series of five problems on duties, graded for class use and based on present tariff schedules. Illustrate the three types — *ad valorem* alone, specific alone, *ad valorem* and specific.

CHAPTER XXI

METHOD AS DETERMINED BY TYPE OF LESSON — DEVELOPMENT AND DRILL

CLASSIFICATION OF LESSON TYPES

Factors determining methodology. — Methodology in teaching depends upon what is taught, who teaches it, and to whom it is taught — upon subject matter, teacher, pupil.

The purpose of this and the following chapters is to discuss the essential principles of method in teaching arithmetic, as determined by *the nature of the subject matter taught*, and by *the purpose of its presentation* — i.e., to discuss the *types* of lessons that occur in arithmetic, with a view to assisting the teacher in skillful presentation.

Method as determined by purpose. — Matter may be presented to the child (1) for the purpose of acquainting him with something new, giving rise to the *development or instruction lesson*; (2) for the purpose of fixing something in mind, giving rise to the *drill lesson*; (3) for the purpose of determining to what extent the pupils have mastered a given body of subject matter, giving rise to the *test lesson*; (4) for the purpose of creating a liking for the subject matter, a pleasurable interest in it, and an appreciation of its values, giving rise to the *appreciation lesson*.

Method as determined by nature of subject matter. — The ultimate goal of arithmetic teaching is to enable the

child to solve *problems*. In solving problems, certain *processes* must be performed, and certain *terms* referred to must be understood. Sometimes the performing of the operations involved is greatly facilitated by familiarity with underlying *rules* or *principles* upon which those operations, or the relations involved, are dependent. Any topic in arithmetic, therefore, falls under one of the four kinds of subject matter :

1. Terms — definitions or meanings.
2. Rules and principles.
3. Processes or operations.
4. Problems or concrete applications.

In connection with any of these four kinds of subject matter a lesson may have for its purpose *instruction*, *drill*, *test*, or *appreciation*. We shall consider first the development lesson for each of the four types.

DEVELOPMENT LESSONS

TERMS

Definitions and meanings. — A term is defined for the pupil if he knows exactly what it signifies, and uses the term accurately when occasion arises. This is of the utmost importance, for if the meanings of the technical terms employed in arithmetic are not definitely known, the task of explaining processes and problems is made greater. Some terms should be defined formally and the definitions learned; others do not require formal definition.

Teaching terms through practical situations. — Every term has arisen in response to some human need. When a

new term is taught, that method of presentation which will cause the pupil to *realize the situation* that gave rise to the term in question will make the most vivid, the most informing, and the most lasting impression.

In this connection a knowledge of the history of the subject will greatly assist the teacher to present it in such a way as to cause his pupils to feel the original need which led to its discovery. For example, if he understands the need which led the people of the world to originate the *foot* measure, he may place his pupils in a situation that will cause them to feel the same need by asking several pupils to measure the length of the room with their feet. The diversity of answers will show the need for a standard unit to replace the indefinite foot. Likewise, if the teacher knows that the standard foot was derived from the human foot, he will be able to invest the study with a richer meaning and interest.

In teaching such terms as commission, discount, etc., the pupil should be helped to realize the situation through the imagination by means of a vivid presentation and concrete illustrations. These subjects can be taught better by one who is familiar with the situations and practices in which the terms are employed than by one whose knowledge is limited to what he may have read in a school text.

It is often helpful in giving the name of the new term to tell why it was so called. A study of the dictionary throws helpful light on most of these names.

Teaching terms through logical division. — In response to the need felt for distinguishing varieties, some terms have been subdivided into two or more different kinds.

That act of thought by which the mind classifies or discovers *kinds* or *species* of a term is called *logical division*.¹ Let us discover the steps in the process by studying instances of its use. On page 137 we are classifying integers. This original term, our starting point, is called the *genus* (*gigno* — the same root that we find in *beginning* — *to be born*). We are interested in determining whether numbers are resolvable into factors; this guiding thought or question which we have in mind is called the *basis of classification*. We find that there are two possibilities, or differences — some numbers have factors other than self and one, and some do not; this step is called *finding the differences*. We have thus found two different kinds or species of numbers which it is convenient to name; so our next step is to *state and name the species*. Thus, numbers which have no other factors than self and one are called *prime*; those which have other factors are called *composite*. Lastly, having discovered and named the new terms, it remains to state precisely what each is — that is, to *define the species*. This consists in stating the genus and the difference: A *prime* number is a number which has no other factors except itself and one; and so for the other term. We may trace the same steps in classifying fractions (p. 159). The complete process of logical division is therefore summed up in the following steps:

1. Fix upon a basis of classification.
2. Discover the differences.
3. State and name the species.
4. Define the species.

¹ BAILEY. — *A Handy Book in Teaching Arithmetic*, pp. 2-7.

When the method of logical division is employed, the classification should be made complete. That is, all the species of the genus should be developed, named, and defined.

It is evident that this is a teacher's tool. If he has these steps in his own mind, he is better able to direct the pupils' line of thought in such a way that they will discover for themselves the different species, and formulate their own definitions. The teacher must supply the question to be answered, *i.e.*, the basis of classification, and also give the names of the species. The pupils will be able to take the other steps, but assistance should be given in formulating definitions accurately.

Use of logical division in organizing subject matter. — Logical division is also helpful in discovering types or kinds of examples or degrees of difficulty in an operation. *Division examples* (the genus) are classified according to the *number of figures in the divisor* (basis of classification) into *short division* and *long division* (names of species).

On page 207 we are discovering the types of examples in *multiplication of decimals* (genus); the difficulty here is the *term in which the decimal occurs* (basis of classification). The differences are given and the species stated on page 207. The species have no names. On page 131 four different bases of classification are used in working out the various difficulties in long division.

Cases or types that arise from considering an equation as our genus may also be treated as an application of the process of logical division. Here the basis of classification is the omission of a term. Thus, on page 51 the equation $6¢ + 4¢ = 10¢$ is considered as the genus. The differences are: the omission of 6¢, of 4¢, and of 10¢.

Stating and naming the species, we have: $? + 4¢ = 10¢$, and $6¢ + ? = 10¢$ (examples in subtraction), and $6¢ + 4¢ = ?$ (an example in addition). On page 176 the original statement of fact corresponds to the genus, the omission of a single term is the basis of classification, and the resulting species are three types of problems, A, B, and C.

Value of recognizing different types. — The discovery of types of examples is a great aid to the teacher (1) in arranging subject matter which is to be presented in order of progressive difficulty, and (2) in recognizing that each type has a meaning different from each of the others, and hence needs a different presentation. Thus, in teaching multiplication by a number of one order we go back to the fundamental meaning of multiplication (p. 97), but in teaching the multiplication by a number of two orders we separate the multiplier into two parts and multiply by each part separately (p. 98).

Again, in considering types of examples in multiplication of fractions, we discover that we may have a fraction in the multiplier, in the multiplicand, or in both terms. This sets the teacher to thinking. Does $\frac{2}{3}$ of 9 (fraction in multiplier) mean the same as $\frac{2}{3}$ multiplied by 9 (fraction in multiplicand)? Indeed not. Then it should not be taught as meaning the same; in the presentation, each should be interpreted according to its meaning, even though for practical purposes the mechanical procedure may be the same.

RULES AND PRINCIPLES

The inductive method. — The thinkers of the world have discovered and established general principles or

truths by two methods of reasoning. One method is based upon observation and experiment. An outstanding truth common to several members of a class is noted; it is then assumed to be true of *all* members of that class. This is known as the inductive method. It is the method of the beginner, and was the earliest method of reasoning employed by the peoples of the world in discovering general truths or principles.

One of the chief laws of the inductive method is the following: *Whatever is true of several individuals of a class is probably true of all the individuals of that class.* This is known as *the canon of agreement*, and forms the guide for our procedure in discovering truths inductively. That which is found to occur in the several instances examined we may call the *common circumstance*.

In using this tool the teacher must choose several particular cases for investigation. He should select typical and simple representatives of the class. The pupil should be encouraged to discover the common circumstance and to state the probable truth for himself.

Illustration: On page 167 the task is to discover how to multiply a fraction by an integer. Here our "class" is *multiplication of a fraction by an integer*. The cases chosen ($\frac{2}{9}$ multiplied by 3 and $\frac{3}{8}$ multiplied by 2) are simple and typical because they serve to disclose the two possible results. The teacher leads the pupils to see that in the result of each case examined the numerator has been multiplied or the denominator divided. The pupil then states the probable rule.

The deductive method. — *Explanation.* — In this method we have a series of related statements each of

which is based upon an accepted definition, axiom, or previously proved proposition. It was originated about 600 B.C. by Thales of Miletus, who is sometimes called the father of the deductive method.

This is a method which a pupil cannot intelligently use until he has a sufficient background of definitions, axioms, and previously proved rules to appreciate their use as the reasons upon which the new steps depend. If this necessary background has no meaning for him, the new steps will be meaningless and the whole procedure then becomes a senseless repetition, distasteful and dreary in the extreme.

Steps. — Let us discover the steps in this method of procedure. Turn to page 169. Here our task is to discover how to multiply a fraction by a fraction. The subject to be investigated is “multiplication of a fraction by a fraction.” In geometry, if our subject of investigation is a triangle, we take a specific triangle and begin by saying, “Let ABC represent any triangle.” We might have begun in a similar way by saying, “Let $\frac{2}{3}$ of $\frac{4}{7}$ represent the multiplication of one fraction by another.” Or, we may assume this to be understood and begin our demonstration at once. First, we state that $\frac{2}{3}$ of $\frac{4}{7}$ is equal to another expression and cite the reason why. The second expression $[(\frac{4}{7} \div 3) \times 2]$ we state to be equal to another and cite the reason why. Our last expression $(\frac{4}{21} \times 2)$ we state to be equal to another and give the reason. From this series of statements we draw our logical conclusion $(\frac{2}{3} \text{ of } \frac{4}{7} = \frac{8}{21})$ from which we formulate the rule. Hence, in general, the technique of the deductive method is as follows: The subject under investigation is

stated equal to some other term or expression, because of a previously established or accepted truth ; this second term or expression is known to equal some other, because of another previously established or accepted truth. A series of statements is thus obtained in the form of a logical syllogism or sorites as shown below :

SYLLOGISM	SORITES
A is B , because	A is B , because
B is C , because	B is C , because
$\therefore A$ is C .	C is D , because
	$\therefore A$ is D .

NOTE. — The *sorites* may have any number of statements. The predicate of each successive statement becomes the subject of the next statement.

Each statement in the syllogism or sorites, except the last, is a pure deduction from a more general truth.

The full formal procedure as shown above may often be abbreviated, but the essential feature must be preserved in any piece of deductive reasoning — namely, that the truth of the new principle depends upon axioms, or accepted definitions, or previously proved propositions.

Uses. — The deductive method may be used not only to establish a truth which has been previously discovered inductively but also to *discover*¹ *new truths*. The notion seems to prevail in some minds that the deductive method is not a tool for discovering new truths. This is a mistaken

¹ Overman in his *Principles and Methods of Teaching* (*op. cit.*) also discusses this matter, but is guilty of a weak point in his development in step (c), p. 76. A truth established by reasoning cannot be *verified* by induction in the sense of establishing its *validity*; it is simply a case of applicability. His comparison of the inductive and deductive methods on page 83 is excellent.

notion. The rule need not be previously stated and then established; it may not be known until the end.¹ The propositions in demonstrative geometry are usually stated at the beginning and then established, but it is not necessary that this should be done.

Deduction is a more advanced method than induction for the reason that each step is established rationally, and the conclusion reached is certain rather than merely probable.

The teacher may adapt this method of proof to the child. The first step is to make sure of the apperceptive basis. The other steps may then follow in an informal way. Suppose the teacher desires to teach the multiplication of a fraction by a fraction. She may proceed somewhat as follows: (1) Review the multiplication of a fraction by an integer and the division of a fraction by an integer in cases like the following: $\frac{2}{3}$ multiplied by 4, and $\frac{5}{8} \div 2$, making sure that the child understands the principle that applies to each. (2) What does it mean to take $\frac{2}{3}$ of 12? Yes, we divide 12 by 3 to find $\frac{1}{3}$ of it, and then multiply this answer by 2. What do you think it would mean to take $\frac{2}{3}$ of $\frac{4}{7}$? Yes, divide $\frac{4}{7}$ by 3 and then multiply the answer by 2. Let us write $\frac{4}{7} \div 3$ and see whether you know how to do this. That is right — we multiply the denominator. Our answer is $\frac{4}{21}$. What part have we now found? $\frac{1}{3}$ of $\frac{4}{7}$. Yes, what must we do now? Find $\frac{2}{3}$ of it by multiplying our answer by 2. Let us write this also: $\frac{4}{21} \times 2$. Do you know how to find this answer?

¹ The principles discovered by Einstein have been worked out deductively. The discovery of the planet Neptune is another illustration of a deductive procedure. The discovery of Neptune is referred to in the *American Mathematical Monthly* for January, 1925.

That is right — multiply the numerator. Our answer now is $\frac{8}{21}$. That is, $\frac{2}{3}$ of $\frac{4}{7} = \frac{8}{21}$. Look at your answer and see whether you can tell me an easier way of getting it. Good. Multiply the two numerators to get the new numerator and the two denominators to get the new denominator. That is an easy way, then, of multiplying two fractions. We call this easy way a *rule*. Who will state the rule? (3) Let us now use this rule in working some examples.

PROCESSES

The arbitrary method. — In teaching processes the teacher has his choice of two methods of procedure, the arbitrary method and the rational method. In the arbitrary method, no attempt is made to have the pupil understand the meaning; he is told *how* to proceed but not *why*. Enough practice is then given to mechanize the procedure. Thus, in teaching the multiplication by a number of two orders, as 24, he is told to multiply first by 4 and then by 2, is shown where to place the right-hand figure of each partial product and is then told to add. He has no conception of *why* he proceeds thus, and neither partial product has definite meaning for him. But by and by he learns *how* to multiply, and this knowledge suffices for practical purposes.

Superiority of the rational method. — The rational procedure, in which the teacher endeavors to have the pupil understand *why* he takes each step, is a much higher order of teaching. The pupil by this method may be able to see what is to be done next, and may in a sense feel that he is a discoverer. This is sure to enrich the work in many ways.

The pupil is more *interested* in something he understands; the fact that he is able to discover things for himself gives him confidence and is a spur to other investigations; and less practice is needed to become proficient in performing the operation.¹ The teaching of operations upon integers, fractions, and decimals has been so fully discussed in Chapters I–VIII that the reader is referred to these chapters for illustrations of rational presentations.

The teacher must recognize the fact that not every child will profit in the same way by a rational presentation. The procedure is virtually the same as the procedure for the deductive proof of a rule. Hence, each child will profit to the extent that he is able—in proportion to his understanding of the previous knowledge upon which the new depends. The fact that not all understand the presentation is no reason why the method should not be used. Those who are able to profit by such a procedure are entitled to the benefits which unquestionably accrue therefrom, and to offer them the mere mechanics would be unfair. Such pupils will be developed along lines that make for real mathematical thought in a way that could not possibly result from a mechanical presentation. Those who do not understand the rational procedure are in no wise injured, as they still have recourse to mechanizing the process in the drill lessons which must follow any development lesson. Since rationalization of a process takes more time in the early stages, the slower children get a better chance to master the mechanics.

¹ "The following aims are of the greatest importance: a progressive increase in the pupil's understanding of the nature of the fundamental operations and power to apply them in new situations." *Reorganization of Mathematics, Bulletin 1921, No. 32, p. 5.*

PROBLEMS

Determining factors of good problem teaching. — Perhaps no other topic in the subject of arithmetic is so poorly taught as the problem work. Good problem teaching depends primarily upon two things: (1) suitable problem material, (2) the teacher's mastery of the technique of problem solving.

Too much emphasis cannot be laid upon the value of good problem material. First of all, problems should be interesting. A child may be interested in a problem because it helps him in something he wants to do, because he sees how it would be a help to somebody in some real situation, because it starts new ways of thinking about his immediate environment or mode of living, because it sheds light upon some interesting subject near or remote, because it arouses wonder or curiosity, because it possesses a puzzle or game interest, or because it requires some activity such as measuring or using objective material, as in playing store, building doll houses, or making furniture for a doll house.¹

Problems should also be true to life and should be within the comprehension of the child. No attempt is here made to discuss fully this important phase of the subject. Much that is worth while is already in print. Thorndike's contribution along this line is one of the finest he has made towards enriching and vitalizing the subject of arithmetic. His strong plea that problems should teach a pupil "to think and to apply arithmetic to situations such as life may offer in useful and reasonable ways" is accentuated

¹ BONSER, F. G. and MOSSMAN, L. C. — *Industrial Arts for Elementary Schools*, pp. 213-215; The Macmillan Company, 1923.

by his many splendid illustrations to accomplish this very thing.¹

Technique. — The technique of problem solving is fully explained in Chapter X. We see that four major things must be mastered: (1) the solution of simple problems, (2) the solution of complex problems, (3) written solutions, and (4) graphic solutions. For full discussions of each see Chapter X.

In solving simple problems we do not wish the work to become mechanical; hence set or formal explanations should be avoided. The writers on page 272 have spoken of the value of the conclusion analysis. To lead up to this form one may become mechanical to the extent of applying this slogan, *Begin your explanation with your required term and show how you get it.*

A child who has been taught a form like the "model" analysis (p. 251) is more likely to depend upon verbal "cues" than upon a study of the relations. Thus, a pupil who repeats mechanically "Since 3 yards of cloth cost \$18, 1 yard will cost $\frac{1}{3}$ of \$18," is apt to explain a similarly worded problem in the same way, "Since it takes 15 minutes to saw a log into 3 pieces to saw a log into 1 piece will require $\frac{1}{3}$ of 15 minutes or 5 minutes" — a statement which is manifestly absurd.

To fasten attention first upon what he wishes to find, and next upon how he is to get it, requires more than mechanical repetition. He must do some thinking of his own before he speaks, and usually that insures the correct solution.

¹ THORNDIKE. — *New Methods in Arithmetic*, pp. 5-12.

OVERMAN. — *Principles and Methods of Teaching*, pp. 240-255.

Complex problems. — Here the teacher wants to help the pupil to form the habit of thinking: What must I find first? What next? What next? and so on; in other words, to gain the power of seeing the different simple problems into which the original problems may be separated. The analytic and synthetic methods of procedure (p. 276) should both be thoroughly understood by the teacher in order that she may the more skillfully direct the pupil's thought. But whichever method is used, the most helpful thing is to have the pupil form the habit of doing but *one thing at a time* and not to feel that he must understand how to solve the whole problem at once. If this habit is patiently cultivated, the result will be a surprising growth in the ability to solve problems.

A great variety of problems should be used, so that the pupil will have practice in solving many different types; but no attempt should be made to mechanize a type, except in cases where the solution of the problem consists in the application of a rule. In such cases, the work becomes more or less mechanical.

Problems in written form (p. 254). — Here again, if the teacher himself is master of the technique, he will stress *essentials* in presenting written problems for first lessons, thus starting the pupils to form good habits in the beginning. The following points should be specially stressed: (a) *Form the habit of abbreviating your terms as you write them.* Unless this is carefully observed the pupil feels it a waste of time to write down all that is given and required. But if he learns to abbreviate, the work does not seem so tedious, and eventually the habit of getting the data before him in clear and concise form is established.

This lays a sound foundation for later work in formulas and equations. (b) *Put away your book as soon as this is done.* This is most important. If the child continues to look in his book for the given data, half the advantage of writing the terms is lost. But if he feels that the book has to be put aside and he cannot refer to it again he will try the harder to make sure of getting all the data before him. This very effort will help him to fix the problem in mind and to understand the situation that faces him. This is a long stride toward victory. (c) *Be neat and accurate in the computations.* (d) *Prove the result.* A justifiable feeling of satisfaction and of self-reliance results from work neatly and correctly done.

Graphic solutions. — To solve problems graphically with success depends upon the ability to picture the relations mentioned in the problem. One important thing to stress here is the choice of that term which serves best as the *basic* term. Thus, if a problem in *profit and loss* is given in which the profit is reckoned on the selling price, the initial line should be the selling price and all the others should be studied with reference to that basic line. If the profit is reckoned on the cost, then the basic line should represent the cost.

This method of solution is so valuable in mensuration problems (see Chapter XIII), in longitude and time (p. 331), in thermometer problems (p. 332), in cases 3, 4, and 5 of fractions and percentage (p. 396), and in many other problems, that it should by all means be introduced and frequently used.

The study and interpretation of statistical graphs and other graphs also properly belongs in the problem field

and may be used in the upper grades. (See Chapter XI.)

Summary. — The relations of the topics discussed in this and subsequent chapters may perhaps be more clearly seen from the following summary in diagrammatic form :

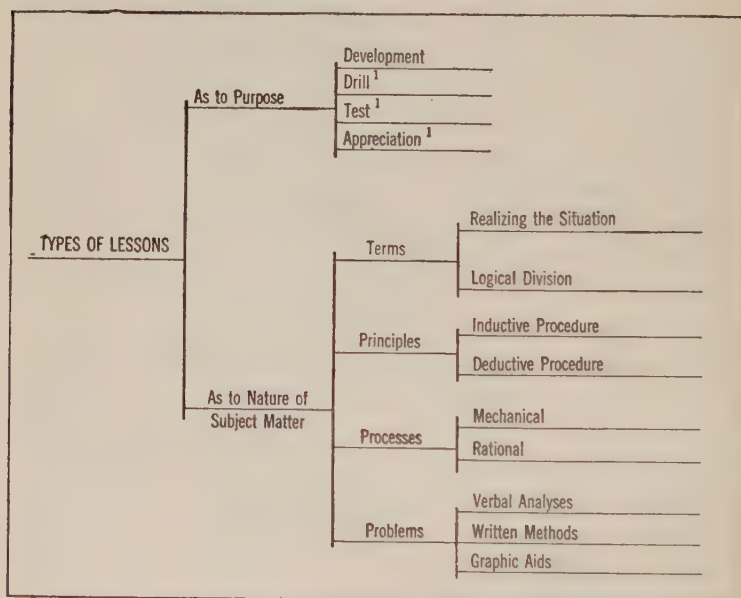


FIGURE 104

EXERCISES

1. Select from Chapters I–XIII a list of twenty-five terms which are developed by *realizing the situation* and discuss the needs that gave rise to each.

2. In the same way list as many terms as you can find that are developed by *logical division* and give the four steps in each development.

¹ Discussed in detail in the remainder of this and in subsequent chapters.

3. Discuss the application of logical division on pages 104, 167, 188, and 191.

4. Make a list of ten principles taught inductively and discuss the procedure.

5. Make a list of ten principles taught deductively and discuss the procedure.

6. Discuss the teaching of (a) column addition (p. 46); (b) column subtraction by Austrian method (p. 68); (c) multiplication by multiples of 10 (p. 98); (d) short division (p. 126); (e) long division (p. 128). Tell whether the procedure is rational or arbitrary, giving your reasons.

7. (a) Make a good list of twenty-five simple problems, five for each of the five operations. (b) Show how you would proceed with each to help a pupil who could not solve the problem.

8. Compile a list of good *real situation problems* (a) suitable for 5th year; (b) suitable for 6th year; (c) suitable for 7th or 8th year.

9. Compile a list of problems to be solved graphically, and discuss what you would select as the *basic line* and why.

10. Make a collection of interesting puzzle problems or mathematical curiosities and tell for what grade each is suitable.

DRILL LESSONS

GENERAL PURPOSES

Statement of purpose. — We have seen that the purpose of a development lesson is to present new subject matter, and the nature of the matter determines the method employed. The purpose of a drill lesson also determines its methods of presentation. Drill lessons have a twofold purpose: (1) To fix something in mind — chiefly definitions or meanings of terms, statements of rules and principles, and certain fundamental facts; (2) to acquire skill in the use of mathematical tools and in performing mathematical exercises — both

abstract processes and problems, and to mechanize certain skills.

FIXING IN MIND

What to fix in mind. — We thus see that the drill lesson covers the same four types as the development lesson — terms, principles, processes, and problems.

In considering drills for fixing in mind we are interested primarily in the *what* and the *how*. The following are representative of content that needs to be fixed in mind :

The combinations in the fundamental processes; the common tables of measures; the decimal equivalents of business fractions; definitions or meaning of common terms; and rules for performing operations in fractions and decimals, and for solving problems in mensuration. It is the belief of the writers that time devoted to memorizing certain important definitions and rules — always on the supposition that the pupil *understands the meaning* of what he memorizes — is time well spent. If we teach these terms and rules at all, we teach them for *use*. We should bear in mind that the formal statements themselves are not ends but means — useful only as they serve the purpose of clarifying the child's notions and as they are in mind when needed.

The long experience of the authors with students who are totally unable to state accurately a definition, a rule, or a principle, or to apply a principle to a case out of the ordinary, leads to the conclusion that there is some virtue at least in the old-fashioned memorizing — all good arguments to the contrary notwithstanding. Great care should be taken, however, to have only *accurate* statements memorized. Far better that no

statement be memorized than that an inaccurate one be remembered.

How to fix in mind. — We all know from experience that some facts are indelibly impressed upon our minds with but one occurrence, while others, even with many repetitions, are not remembered. We remember best those things which grip us — upon which our whole attention is concentrated. So it is in drill exercises: a few repetitions when pupils are interested, alert, and attentive are worth a dozen dull, monotonous drills when pupils are weary or listless. In other words, *repetition with satisfaction or pleasure*¹ is in general the answer to our question.

Some specific suggestions may prove helpful.

(a) *Vary the drill.* The younger the child the sooner he tires of one thing. The same thing must be repeated in various ways. No one form of drill should ever be used for a long period. Hyde says in discussing methods for primary grades, "Give a child four lessons to *learn* in a school day and he will learn next to nothing, give him twenty different things to *do* each requiring from ten to fifteen minutes and you will have very little trouble with him and he will learn a great deal."² Variety in the practical uses made of new ideas also assists in fixing them in mind.

¹ Thorndike states the laws for the formation of mental connections thus: (1) The law of exercise: "Other things being equal, use strengthens and disuse weakens mental connections." (2) The law of effect: "Other things being equal, connections accompanied or followed by satisfying states of affairs are strengthened, whereas connections accompanied or followed by annoying states of affairs are weakened." THORNDIKE, E. L. — *New Methods in Arithmetic*, p. 57.

² HYDE, WILLIAM DEWITT. — *The Teacher's Philosophy*, p. 6; Houghton Mifflin Company.

(b) When intense application is needed *make the drill short and snappy* — from two to ten minutes according to age of pupils and character of work.

(c) Spend time on what a child has not mastered, not on what he already has in mind. In other words, *use selective drills*.

(d) Keep *all* busy *all* the time. This is a difficult thing to do, but nearly all drill exercises should provide for sifting out those who are expert from those who need more practice. Those who do not need further drill should not be stultified by having to sit and listlessly listen to the others but should be given other work while those who need more drill are receiving it.

Exercises from some book other than the text in use, devices for seat work put out by any of the companies dealing in educational supplies or by well-known publishing houses, are valuable for those who do not need the additional drill. Extra credit should be given for completed units of such work.

(e) *Avoid nervous haste*. — Poise in a drill lesson is just as admirable and essential as poise in character. One who hurries usually does a thing poorly or forgets and must do it over. Hurry distracts.

(f) *Do not dawdle*. Hurry confuses; dawdling bores.

(g) *Rhythmic exercises* are sometimes helpful. Used for individuals they often aid in concentration. Used in concert for two or three minutes they sometimes wake up a listless class. Concert exercises, however, should be used but rarely, and always for the purpose of helping individuals, by amusing, encouraging, or setting a standard.

(h) *Use devices* that appeal to the eye and to the ear.

Concrete material should be used in drills as in development lessons. Thus, in teaching the combinations in addition, a considerable amount of drill work is required before the pupils will be able to call sums immediately upon their audible or visual presentation. If number pictures have been used in the first step, the pupils will, for a time, make use of their mental images of these pictures, but eventually they will get away from the pictures entirely. $5 + 2$ or $\frac{5}{2}$ comes to represent *seven* in the same way that the word *dog* represents the idea, all conscious reference to, or dependence upon, counting or imagery being absent. Charts, textbooks, perception cards, practice sheets, are all valuable in drill exercises and should be employed. But the material should not be so interesting in itself as to distract the mind from the mathematical content which it is designed to fix in mind. Thus, if flash cards of brilliant colors were used, the child might become so absorbed in the color as to become inattentive to the combinations.

(i) *Something put in story form or verse* is often more readily held in mind. "Thirty days hath September, etc." is a good illustration. A little attention to dressing up facts in this form would yield a surprising return.

(j) *Dramatization* is a form of the story that makes a strong appeal to the child. The simple device of personifying a dreary entity such as an improper fraction by beginning, "One day Miss Fraction looked at her numerator, and saw it was larger than her denominator," etc. will do more to help fix troublesome terms in mind than many ordinary repetitions.

ACQUIRING SKILL

Effect of practice. — *Need.* — Let us now consider the second purpose of drill — that of acquiring skill in the use of mathematical tools and skill in performing mathematical exercises. The only way to become skillful in doing a thing is to do it. A person may know perfectly all the movements for swimming by watching another swim or by having some one explain these movements to him, but he never learns to swim until he gets into the water and *uses* his knowledge. A person may know all the musical notes perfectly as he sees them in print and may know the corresponding keys to strike on the piano, but this knowledge is not sufficient to make a skillful pianist. The laws of habit formation¹ are the laws that apply in acquiring skill.

Results obtained. — That practice yields results has been recognized by teachers throughout the entire history of teaching, but only recently have we been learning *how much* improvement may be expected from a given amount of practice in a given skill, how to make the best use of time devoted to practice, and how to secure the most favorable mental attitude toward practice on the part of those who are to engage in it. There has grown up during

¹ "The older psychology, perpetuated in current educational doctrines and practices, regarded reasoning as a force largely independent of associative habits, which worked back to correct or oppose them. Our present psychology finds that the mind is ruled by habit throughout, the correction or opposition being of certain more simple, thoughtless, and coarse habits, by others which are more elaborate, selective, and abstract. It defines reasoning as the organization and coöperation of habits rather than as a special activity above their level; and expects to find "reasoning" and habit or association working together in almost every act of thought." THORNDIKE, E. L. — *The Psychology of Algebra*, p. 458; The Macmillan Company.

Overman, J. R., also discusses habit formation in his *Principles and Methods*, pp. 136-141.

the past few years a rather extensive literature, treating of this phase of the teaching process.

J. C. Brown made a study of the value of drill¹ on the four operations. Three classes — sixth, seventh, and eighth grades — were each divided into two groups of equal size and ability in written operations. Groups number I were given five minutes daily drill on the fundamental operations, while groups number II were given no formal drill. After thirty recitation periods in which both groups of each class did the same work (apart from the drill) under the same teacher, tests similar to those used in determining the groups were given; these tests came at the end of the school year. A third test was given at the end of the twelve-weeks vacation. The results are summarized in the following table.

	DRILL GROUPS			NON-DRILL GROUPS		
	1st Test	2d Test	3d Test	1st Test	2d Test	3d Test
Medians	41	53	52	41.5	49.5	46
Av. Dev.	12.6	12.6	10.64	9.61	9.61	11.53

It is seen that the drill groups show about 50 per cent more gain than the non-drill groups over the period of six weeks, and their permanent gain (comparing 3d test with 1st test) is 11 points as against 4.5 points for the non-drill groups — a superiority of 140 per cent. A second study with 222 sixth-grade children, conducted in similar fashion, yielded the same kind of results.

¹ BROWN, J. C. — "An Investigation of the Value of Drill Work in the Fundamental Operations of Arithmetic"; *Journal of Educational Psychology*, November and December, 1912.

Hahn and Thorndike's investigation. — Hahn and Thorndike made a similar investigation,¹ confining the work to addition. They conclude: "It seems therefore safe to say that pupils, taken as they are found, will gain nearly one per cent a minute in amount done correctly from an hour's distributed practice in adding, and will maintain or improve the percentage of correct answers in their total product." It was found that the gains made by those of superior initial ability were greater in *amount* than the gains by those of inferior initial ability; but the gains expressed as per cents of initial ability are greater in the case of those of inferior initial ability.

Donovan and Thorndike's experiment. — Another experiment² measured the effect of two-minute drills given twice a day for 15 days — total practice time one hour. The work was confined to adding single columns of ten figures. The conclusion is thus stated: "The group as a whole improved . . . from an average score of $2\frac{3}{4}$ examples correctly done per minute in the first two periods, to a score of $4\frac{1}{2}$ examples done correctly per minute in the last period. The results thus emphasize the very great gain probably to be expected from applying the method of the practice experiment to certain functions whose improvement is a part of the school curriculum." The average gross gain of the twenty-five per cent of the class that stood lowest in initial ability was just a trifle less than for the twenty-five per cent that stood highest in initial study.

¹ HAHN, H. H. and THORNDIKE, E. L. — "Some Results of Practice in Addition under School Conditions"; *Journal of Educational Psychology*, February, 1914.

² DONOVAN, M. and THORNDIKE, E. L. — "Improvement in a Practice Experiment under School Conditions"; *American Journal of Psychology*, XXIV, p. 426.

Kirby's study. — A study by T. J. Kirby¹ of the results of 75 minutes of distributed practice by 732 fourth-year children shows large gains; in *addition* a gain of 48%, in *division* a gain of 75%.

Conclusions. — These experiments point to the important conclusion that practice not only develops skill but practice periods so managed that the pupils know what their scores are as compared with others or with established standards, and are kept advised of their improvement, are much more effective than the same periods not managed in some such way. In other words, the time element in drill seems relatively unimportant as compared with the way the work is conducted.²

Formation of good habits. — Many of the suggestions given in the preceding paragraphs on *fixing in mind* apply with equal force to the *acquisition of skill* in performing mathematical exercises.

In assisting pupils to form good habits, it is helpful to:

1. Analyze the causes for poor work and use the remedy that fits the disease. Accuracy in written work in the fundamental operations requires a complex set of adjustments including such elements as accurate visualization of the numbers in the algorism, accurate motor response, accurate recall of the combination needed, writing of the right figure.

The teacher must take these things into account, and

¹ "Practice in the Case of School Children"; *Teachers College Contributions to Education*, 1913.

² That the amount of time is less important than the use made of it in all arithmetic work is proved by Rice's study. He found that there was no correlation between achievement in arithmetic and amount of time devoted to the subject, either during school or in home work. — See "Educational Research: Causes of Success and Failure in Arithmetic"; *The Forum*, XXXIV, pp. 437-452.

must so organize and graduate the drill work that the children may not be discouraged by failure. It is important that the child's work be so arranged as to minimize the chance of seeing the wrong thing, and that the examples be made very short and easy until the arrangement of the work and the motor elements have become more or less automatic.

2. Insist on use of working explanation. A cause of poor or slow work may be that the child is saying so much to himself as he works that his procedure is necessarily slow and laborious. The child needs practice on the short *working explanations* (or "inner speech") so often referred to in this book. A lengthy form of inner speech as he works tends to dissipate rather than concentrate attention upon what he is doing, and this seriously interferes with acquiring skill in computation. The use of "crutches" also interferes with a skillful execution of a piece of work.

3. Insist on correct performance of the task from the first. The habit of doing a piece of work in a slipshod or careless manner is easily formed, and should be broken up at once as it is a great hindrance to an efficient performance of a task.

Motivation of practice work. — Hyde ¹ says that in the intermediate and grammar grades the child is at an age when natural impulses lead away from those things of ultimate worth. It is the teacher's task here "to lighten and brighten the larger good with such artificial encouragements, rewards, advantages, and attractions that, in spite of the pull of nature in the opposite direction, the child will choose that larger good which arithmetic . . . and

¹ *The Teacher's Philosophy*, p. 15.

the other studies represent. It is to weight down restlessness, and inattention, and laziness and inconsiderateness of all sorts with such automatic artificial penalties, and privations and disabilities, that he will of his own preference reject them."

Games and contests. — In lightening and brightening the arithmetic drills, games and contests are especially useful.

In the intermediate and grammar grades, particularly, contests of various sorts add a surprising stimulus. So much good material has been written on this subject in recent books that the writers prefer not to attempt a lengthy discussion of the subject. They know from experience that a contest between two well-balanced groups for a series of weeks secures results beyond belief unless one has tried it. Leaders coach up backward members on their side. Errors usually familiar to teachers only, are recognized by all and some members are always ready to set about helping to correct them. In such a contest, particularly when the points are won by means of the relay race, accuracy and good form should always be the prime factors in deciding upon the winner. Only when both sides tie on accuracy and form should the time element enter in as a deciding factor. In a contest carried on through a series of weeks, points may also be won by giving oral and written exercises in which the side having the greatest number of correct answers is the winner.

A relay race for a single recitation will often treble the amount of work done by the class as a whole and more than treble the interest taken in the work.

A few illustrative procedures for games and contests

may prove suggestive of many others that the teacher can easily originate :

(a) Competition in reducing per cents — suited to sixth grade. "I will give you a certain per cent; you may write the successive multiples of the per cent I give you; then write the fractional equivalent of each if it reduces to a business fraction, otherwise its decimal equivalent. Each pupil may write three per cents, with equivalents, and each has the right to correct any mistake made by a previous pupil, but no one may have a second turn. Are you ready? 5% — Go!" The first pupil writes :

$$\begin{aligned} 5\% &= .05 \\ 10\% &= \frac{1}{10} \\ 15\% &= .15 \end{aligned}$$

(b) Various exercises with fractions may be profitably used. Here is a suggestive procedure for addition: "I will give you the first fraction, the leader may write it and three more fractions, each having for the numerator, the denominator of the preceding fraction, and a denominator one more than the numerator; the second pupil may find the L.C.D. and write it; the third may write the numerators; the fourth may add and reduce; the fifth may check the work. Each one has the right to correct any mistakes made by a previous pupil before doing his own work. Here is your first fraction: $\frac{2}{3}$ — Go!"

The work appears as follows :

$$\begin{array}{r} \frac{2}{3} \qquad 40 \\ \frac{3}{4} \qquad 45 \\ \frac{4}{5} \qquad 48 \\ \frac{5}{6} \qquad 50 \\ \hline 3\frac{1}{20} \quad 180 \end{array}$$

Such contests serve not only to improve the content of the work but to develop the power to heed definite directions.

(c) Addition relay race. — Each team starts with a different example of any number of addends. The captain adds; number 2 then adds

	43	the whole column; etc. Then the captain divides
	76	the final sum by 2 to the $n - 1$ power, when n equals
	29	the number of players. This should give the captain's
Cap.	<u>148</u>	sum.
2d	<u>296</u>	
3d	<u>592</u>	
4th	1184	
	etc.	
	8)1184	
	<u>148</u>	

(d) Subtraction. — To prepare the exercise the teacher multiplies the number to be used as subtrahend by the number of players on a team.

	4768	Preparation
	<u>7</u>	
	<u>33376</u>	
Cap.	33376	Team work. — Each captain writes this product
	<u>4262</u>	and increases it by a number taken at random. Each
	37638	team has now a different minuend. Each player
	<u>4768</u>	in turn subtracts the announced subtrahend.
	32870	The final remainder should equal the number
	<u>4768</u>	added in by the captain.
	28102	etc.

(e) Addition and subtraction. — For teams of eight.

Preparation: The teacher selects some number, n , and multiplies it by 3. He gives the captains each a different starting number.

Each of the first 6 players adds n and players 7 and 8 subtract $3n$. The final result must agree with the starting number.

(f) Multiplication. — Give each captain a number in which the sum of the digits is a multiple of 9, as 4626. Use in turn any multipliers that are not multiples of 3.

The teacher can check a result by seeing if the sum of its digits is a multiple of 9.

(g) Division. — By keeping a record of the numbers used and the results obtained in the multiplication races, material for division races

may be accumulated. The final product becomes the initial dividend; the multipliers become the divisors. The initial multiplicand becomes the final quotient.

(h) Short multiplication and division. — Give each captain a different number. Each player plays twice, once multiplying and once dividing.

Multiply by 3, 5, 8, 9, 2, 6, 7, 4.

Divide by 9, 6, 2, 7, 5, 3, 8, 4.

Since the multipliers and the divisors are the same, the final quotient will be the original multiplicand.

Games with bean bags may be played in the classroom, and may be varied at will to afford whatever number work the class may need.

Draw two or more concentric circles of sizes suitable to the grade. For first-year children they should be of radii 6, 12, and 18 inches respectively. Write in these circles the numbers you wish the children to use. At a suitable distance from the circles draw a line back of which the player must stand when pitching the bags. The class is divided into teams.

(i) Addition combinations. — Each player pitches two bean bags and announces the combination he makes by naming the addends but not stating the sum. The players record the score by writing the

combination and the sum ($3 + 2 = 5$, or $\frac{3}{2}$), or by writing the sum only.

If combinations having a given constant addend are desired, each player will pitch one bag and combine the number scored with the constant announced. The child, or the team, making the fewest mistakes is the winner.

Two circles will provide work on three combinations, *e.g.*, $2 + 1$, $2 + 2$, $1 + 1$; three circles will give six combinations; or, using the square shown below, any of the 45 combinations may be made. A

2	5	1
8	9	7
4	6	3

zero combination will occur whenever one of the bags falls without the figure.

(j) If practice on the combinations is not desired, the game can be conducted as follows: Each player in the first team pitches one bag, no bag being removed until the umpire has read the numbers on which the bags lie. All write the numbers in a column headed "Team 1." This is repeated with each successive team, after which the totals are found, checked by the teacher, and penalized for mistakes.

It is important to make the drill progressive; each piece of work perfectly done should be followed by something a little more difficult. Work in which a pupil has already attained a high degree of skill does not satisfy him any more than do baby games satisfy him in his play.

Suggestions as to good games and how to use them are found in many recent texts.¹ Game material that may be adapted to the classroom is easily obtainable.²

Devices. — Much excellent practice material may easily be obtained to be used for seat work. The Courtis Practice Tests are well known.³ Lennes and Fowlkes and Goff have sets of practice exercises in book form with accompanying cards for record of progress.⁴ The Maxson cards are excellent for individual work. There are sets in addition, in subtraction, in multiplication, by one-, two-, and

¹ LENNES, N. J. — *The Teaching of Arithmetic*, pp. 141-153.

OVERMAN, J. R. — *Principles and Methods of Teaching*, pp. 180-194.

DAVIS, S. E. — *The Technique of Teaching*, pp. 243-254; The Macmillan Company.

CHADSEY, C. E. and SMITH, J. H. — *Efficiency Arithmetic, Primary*.

² Educational Games for School and Home Play, Cincinnati Game Co., Cincinnati, Ohio.

FLYNN, FLORENCE JAMES. — "Mathematical Games, Adaptations from Games Old and New"; *Teachers College Record*, III.

Ella Crandall Waterhouse has copyrighted "Rith" — a game for addition, subtraction, multiplication, and division.

³ Courtis, S. A., Detroit, Michigan.

⁴ LENNES, N. J. — *Work, Drill and Test Sheets in Arithmetic*; Laidlaw Brothers, Chicago and New York.

FOWLKES, J. G. and GOFF, T. T. — *Practice Tests in Arithmetic*; Macmillan.

three-figure multipliers, and in short and long division. Each of the 60 cards in a set is different from each of the others, but all in any one set are supposed to be of the same degree of difficulty. There is therefore no possible chance of "cribbing." An easy key enables the teacher to correct the answers at a glance.¹

Many simple blackboard devices may also be used for the primary grades to "dress up" a drill in the garb of a game, such as: the combinations written as stepping stones; as picking apples from a tree; as going under an umbrella out of the rain; climbing a ladder; going up stairs and down as fast as possible; uniting as many combinations as will make a given sum in a stated time; writing combinations across the board and having two pupils start at either end to see who can write the more by the time they meet. The ingenious teacher will think of many others.

Importance of accuracy. — Another important thing to bear in mind in computation work is that accuracy is far more important than speed. Absolute accuracy is the essential thing in most practical situations demanding computation, but a high degree of speed is far from essential. To most people it matters little whether they can compute rapidly or not; and where great rapidity is requisite, the computing machine is coming more and more into use. To devote much time in the elementary school to the development of great speed in calculation is evidence of lack of a proper sense of relative values. A reasonable degree of speed will be attained in the process of securing accuracy; but it is to be remembered that

¹ J. L. Hammett Co., Maxson Cards; Boston, Mass.

there are wide individual differences in this respect, and that it is both impossible and undesirable to set anything like a fixed standard of speed to apply to all children. It is known that some children of low general intelligence excel in abstract number work, and that some of high general intelligence rank low in this trait. It is always a mistake to stimulate children to a degree of speed which sacrifices accuracy. It is a psychological impossibility to make young children feel the imperative need for accuracy as an adult feels it. All motives for accuracy must be artificially created. The child, if properly trained, will gradually acquire the desired ideal. If before the end of his school life a pupil has learned the need for accuracy and can work accurately at a rate suitable to himself, the teacher's task in this respect has been well done.

Acquiring skill through problem work. — Problems are not often considered as drill material, but one of the richest fields in acquiring mathematical skills is the problem field. The paragraphs that follow offer suggestions as to how problems may function in acquiring skill.

Puzzle problems, including such historical curiosities as magic squares and magic circles, latticed multiplication (p. 88), and numerical curiosities such as White ¹ mentions may sometimes be used as drill material.

Constructive work or project problems. — The work done in the kindergarten is full of suggestion for primary grades. Some primary textbooks give directions for making toy articles,² such as house, table, chair, bedstead, badges,

¹ WHITE, W. F. — *The Scrap Book of Mathematics*; Open Court Publishing Company.

² HARRIS, A. V. and WALDO, L. M. — *First Journeys in Numberland*; Scott Foresman and Company.

valentines, picture frames, and so on. This develops skill in the use of common measures and in construction of plane figures in common use. All mensuration work — drawing figures and measuring their dimensions, drawing to scale, constructing solids — and exercises in denominate numbers afford rich fields for work with the hands.

Such projects as making articles for sale in playing store and constructing doll houses and furniture for them serve to arouse an interest in, and give meaning to, the daily practice in computation, to say nothing of the excellent opportunity afforded for correlation with other subjects in such a way as to give the project enhanced educational value.¹ Several projects suitable for primary grades are explained in a little book of recent publication.²

If an entire project cannot be carried through, a portion of it may often be used and found to be stimulating and profitable. Instead of working out the entire doll house project the following simple exercise is interesting and gives the child a right idea of proportions.

Ask each pupil to bring a pasteboard corner to represent two adjacent walls of a room. (1) Have him measure the height of his pasteboard and compute the scale he will need for his constructions by finding the ratio of the height of his pasteboard to the height of an ordinary room, such as 9 ft. Thus, if his pasteboard is 9 in. high, his scale will be 9 in. to 9 ft. or 1 to 12. (2) Have him measure the dimensions of a window and door in a room, and then compute the dimensions, distance from floor and from ceiling needed for his model; then cut out the openings — a window in one wall and door in the other. (3) Measure the dimensions of a well-proportioned chair and then have

¹ BONSER, F. G., and MOSSMAN, L. C. — *Industrial Arts for Elementary Schools*; The Macmillan Company.

² McLAUGHLIN, KATHERINE L., and TROXELL, ELEANOR. — *Number Projects for Beginners*; J. B. Lippincott and Company.

him compute the dimensions of a chair of similar proportions for his corner. The work may be done to the nearest 8th or 16th of an inch as desired. (4) A stiff oak tag chair may then be cut, fastened together, and set up as a pattern to show the proper proportion of the chair in relation to the height of room, window, and door. (5) If permanent work is desired, a simple chair of wood may be constructed with same dimensions as the pattern. (6) Other articles of furniture such as a table, davenport, bedstead, etc., may be constructed, using the simplest possible forms, to emphasize *symmetry* and *proportion*.

The subject of bills and accounts also lends itself to the project method. The following procedure illustrates how the topic may be presented.

Bills and receipts. — The use of bills and their essential elements can best be defined through the study of actual forms. A penmanship period might be devoted to the preparation of bill forms for use in the arithmetic period. This will help to fix the form and the parts in mind. The children can use these forms for exercises like the following :

1. The children *play* that they are salespeople in a grocery store or general market. The teacher simulates calling the store on the telephone and asks for Mr. or Miss So-and-so (calling some pupil's name). The pupil asks any questions as to quality and price and all the children take down the order and compute the bill in proper form.

2. The procedure may be varied by having half the class act as salespeople and the other half as customers. A child from each group is selected to carry on the conversation. The first child to finish his bill correctly (the teacher inspects) is permitted to make the delivery. He collects the bills as they are finished, distributes them among the customers, and simulates the delivery of the goods. Each customer checks the bill he receives to make sure it is correct. He then simulates paying in cash, and the delivery man receipts the bill in due form.

The mathematical content of the work on bills needs to be controlled by the teacher so as to bring in plenty of work in fractions. This can be done when the teacher does the ordering, and will lead to figuring the cost of a number of ounces at a price per pound, of a mixed number of yards at a price per yard, the cost of 5 bunches of shingles at a price per M (a bunch contains 250 shingles), the cost of a number of pounds of chicken feed at a price per hundredweight, etc.

EXERCISES

1. Select some topic to be *fixed in mind* and illustrate *variety* in drill work upon it.

2. Explain, or demonstrate, how "flash card" drill on combinations may be conducted so that individual difficulties will receive special emphasis. Make the drill *selective*.

3. Prepare, for each grade, a list of things that would keep the more capable members of the class profitably employed while the slower ones are practicing upon the minimum essentials for the grade.

4. Write down what you would say to a fourth-year class about to take a drill exercise in written multiplication, with a view to getting earnest effort without nervous haste.

5. Prepare a drill exercise in which is stressed the placing of the decimal point in (a) multiplication of decimals, (b) division by a decimal. How would you conduct these exercises?

6. The meaning of a term is more easily fixed in mind if attention is called to the appropriateness of the term. Illustrate how this may be done for each of the following: *improper fraction*, *mixed number*, *odd number*, *even number*, *multiple*, *remainder* (in division).

7. Mention a game suitable for use in each of the grades. Tell how you would conduct the game (a) so as to keep all the children constantly busy, (b) so as to correct inaccuracy due to hurry and carelessness.

8. Show how drill on the "counting series" in addition can be conducted so as to secure the attention of all the children.

CHAPTER XXII

TEST AND APPRECIATION LESSONS

THE TEST LESSONS

Purposes and kinds. — The purposes of tests are (1) to measure a pupil's progress, (2) to reveal to him his own deficiencies and errors, in order that he may set about correcting them, (3) to reveal to a teacher the weaknesses of individuals, and of a class, in order that he may the more skillfully plan his work, and (4) to guide supervisory officers in classifying children, and to measure the efficiency of instruction in order that they may make constructive suggestions for improvement.

In the subject of arithmetic it is comparatively easy to measure the results of class work. There have been placed upon the market a large number of tests more or less standardized. They are, as a rule, quite inexpensive and their use is becoming general in progressive schools. It is important that the teacher should be familiar with these tests so that he may intelligently coöperate with his supervisory officers. Moreover he should be able to use such materials for checking up, from time to time, the results of his teaching.

These commercial tests, however, do not give a teacher sufficient material for the necessary testing of his class for a term's work — the non-standard or individual test must also be used to satisfy completely the first three purposes

of the test, named above. The teacher therefore needs to understand the technique of both the standard and the non-standard test.

STANDARD TESTS

Uses. — Teachers, as well as supervisors, should understand that there are certain very important mental results to be derived from the proper study of arithmetic which have thus far evaded all attempts at measurement; and there is a feeling among experienced teachers that at least some of these results will continue indefinitely to escape the meshes of the measurers.

The writers have no desire to depreciate modern methods of educational measurements, but it is important not only to explain the legitimate uses of standard tests, but also to warn the young teacher against the natural tendency of the enthusiast to expect too much from tests and to draw hasty and unwarranted conclusions from test results. The subject as a whole is rather complex; to escape error one must be versed in the theory of educational measurements and must have had considerable experience in testing.

It is not our purpose here to give a complete treatment of tests in arithmetic. Our purpose is, rather, to describe some of these tests and to explain how the teacher may use them for the purpose of improving his work.

Research or survey tests. — These tests are intended primarily for the use of supervisors and superintendents. They are commonly given at the beginning and at the end of a term of instruction for the purpose of measuring the *improvement* made by the children. Measures of this

sort are highly reliable when applied to large groups of children (say all the children of a city school), less reliable when applied to a small group (as a single class), and quite unreliable when applied to a single individual. That is, if a supervisor finds that the improvement in score in school No. 1 is decidedly greater than in school No. 2, he can safely conclude that the instruction in No. 1 has been superior to the instruction in No. 2 *in the particular aspects of the work which the tests are designed to measure*. If a principal finds that Miss A's class has made a larger gain in score than Miss B's class, he is justified in taking cognizance of the fact, and in seeking the explanation, but this result *alone* is insufficient basis for sweeping judgment as to the relative merits of the teachers concerned. If a teacher finds that one of her pupils does no better on a given test at the end of a term than at the beginning, she is not to conclude that the child has not improved. His mental set toward the second test is only by chance the same as it was toward the first test. The progress of a given pupil can be judged only on the basis of many tests given at intervals throughout the term.

A brief description of some of the better known standard tests is given in the following pages.

The Courtis Standard Research Tests, Series A. —
Time, 24 minutes.

Tests 1, 2, 3, 4 — Fundamental combinations in the
four operations

Test 7 — Written examples in the four operations with
integers

Test 6 — Ability to select the proper operation in simple
problems

Test 8 — Ability to reason complex problems and to carry out the computations accurately

Test 5 — Speed in copying figures

These tests have had a wide use over a period of years. They have been, and still are, of great value for survey purposes. Their chief advantages are: (1) It requires only twenty-four minutes of working time to take the eight tests; the whole series may be given and scored (by the children) in less than an hour. (2) They have been very thoroughly standardized. Their chief defects and limitations are that they do not help the supervisor to locate the specific difficulties which cause low scores. This is especially true of Test 7, which measures a complex of many arithmetical abilities. Mr. Courtis, recognizing this lack, has devised the Series B tests, to take the place of Test 7.

The Courtis Standard Research Tests, Series B. —

Column addition — 8 minutes

Column subtraction — 4 minutes

Column multiplication — 6 minutes

Long division — 8 minutes

This series reveals a pupil's weakness in a particular operation, hence is superior to Test 7, Series A, especially for use by the principal who is interested in locating points of weakness. The value of Test 4 suffers from the fact that all the examples give exact integral answers. This feature flies in the face of the law of probability and gives the child an assurance with respect to his answers to which he is not entitled. Division examples do not happen that way. The following is an illustration:

Measure the efficiency of the entire school, not the individual ability of the few.¹

COURTIS STANDARD RESEARCH TESTS

Arithmetic. Test No. 1. Addition

Series B Form 1

SCORE
No. Attempted_____
No. Right. _____

You will be given eight minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.¹

927	297	136	486	384	176	277	837
379	925	340	765	477	783	445	882
756	473	988	524	881	697	682	959
837	983	386	140	266	200	594	603
924	315	353	812	679	366	481	118
110	661	904	466	241	851	778	781
854	794	547	355	796	535	849	756
965	177	192	834	850	323	157	222
<u>344</u>	<u>124</u>	<u>439</u>	<u>567</u>	<u>733</u>	<u>229</u>	<u>953</u>	<u>524</u>

Two similar groups follow this one, each containing eight examples.

Test No. 2 contains twenty-four subtraction examples like the following :

107795491	75088824	91500053	87939983
<u>77197029</u>	<u>57406394</u>	<u>19901563</u>	<u>72207316</u>

¹ The headings for the other tests are similar to this.

Test No. 3 contains twenty-five multiplication examples like the following :

$\begin{array}{r} 8246 \\ \underline{29} \end{array}$	$\begin{array}{r} 3597 \\ \underline{73} \end{array}$	$\begin{array}{r} 5739 \\ \underline{85} \end{array}$	$\begin{array}{r} 2648 \\ \underline{46} \end{array}$	$\begin{array}{r} 9537 \\ \underline{92} \end{array}$
---	---	---	---	---

Test No. 4 contains twenty-four examples like the following :

$25\overline{)6775}$	$94\overline{)85352}$	$37\overline{)9990}$	$86\overline{)80066}$
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The Cleveland Survey Tests. — Time : 22 minutes.

1. Combinations in the four operations (not complete)
2. Column addition
3. Column subtraction
4. Multiplication by a number of one order
5. Addition and subtraction of fractions
6. Short division
7. Single-column addition — 13 addends
8. Long division by numbers of 2 orders
9. Multiplication by numbers of 2 orders
10. Column addition of 5 addends, each addend of 4 orders
11. Multiplication and division of fractions

The work in the combinations is not complete, but the work of the other tests is satisfactory for the ground that is covered. The median scores for pupils in St. Louis and Grand Rapids are given, showing the number of examples correct for each separate test in the series and for each grade from 3A to 8B. Illustrations of four of these tests are found in the chapters on addition, subtraction, multiplication, and division — pages 25, 52, 78, 108.

The Wisconsin Inventory Tests. — These comprise :

1. The combinations in addition, subtraction, and multiplication.
2. Quotients with remainders within the combinations.
3. Decades in addition.
4. Short division with zero difficulties.
5. Long division — two-figure divisors.

These tests are not designed for survey tests ; hence standards are not emphasized. Each pupil's paper is taken up as soon as he finishes. The pupils are urged to keep individual records, showing just what failures they have made. These inventory tests are to be used for finding just where the troubles are.

The Monroe Diagnostic Tests. — Twenty-one tests.

Sheet No. 1. Time : 8 minutes.

Single-column addition — 3 addends.

Decades in subtraction.

Multiplication by a number of 1 order.

Short division.

Column addition — 5 four-figure addends.

Long division by numbers of 2 orders, two-figure quotients.

Sheet No. 2. Time : 12 minutes.

Single-column addition — 13 addends.

Multiplication by numbers of 2 and 3 orders.

Column addition.

Long division by numbers of 2 orders, five-figure dividends.

Sheet No. 3. Time : $8\frac{1}{2}$ minutes.

The four operations in fractions.

Sheet No. 4. Time : $2\frac{1}{2}$ minutes.

Tests in placing decimal point in multiplication and division of decimals

The purpose of this set of tests is to diagnose difficulties in the subjects covered. The tests contain no combination work. The most difficult long-division examples have only two-figure divisors. The zero difficulties in multiplication are well covered in Test No. 10. The fraction work in the four fundamental operations with simple fractions is very satisfactory, as each operation is tested.

The following excerpts illustrate Sheet No. 1 of these tests :

Test 1 — ADDITION

									At. _____			
									Rt. _____			
4	5	2	0	1	7	6	7	3	2	3	9	
7	5	6	3	1	2	8	7	8	4	3	4	
<u>2</u>	<u>9</u>	<u>7</u>	<u>8</u>	<u>4</u>	<u>3</u>	<u>4</u>	<u>0</u>	<u>9</u>	<u>0</u>	<u>6</u>	<u>5</u>	

A second row similar to the above is given.

Test 2 — SUBTRACTION

								At. _____	
								Rt. _____	
37	94	60	27	39	41	77	53		
<u>5</u>	<u>8</u>	<u>3</u>	<u>6</u>	<u>7</u>	<u>8</u>	<u>3</u>	<u>9</u>		

Two other rows follow.

Test 3 — MULTIPLICATION

					At. _____
					Rt. _____
6572	6750	5863	3754	2845	
<u>6</u>	<u>9</u>	<u>2</u>	<u>5</u>	<u>8</u>	

Three other rows follow.

Test 4 — DIVISION

				At. _____
				Rt. _____
8)3840	4)7432	7)2534	3)8430	6)4680

Two other rows follow.

Test 5 — ADDITION

					At. _____	
					Rt. _____	
8758	2462	1247	4319	6794	3293	7917
2350	9869	3573	2358	5420	7805	4304
3197	4572	1081	5795	4570	7642	9027
2338	6420	7805	4314	8028	7803	9975
<u>5917</u>	<u>6772</u>	<u>9864</u>	<u>1249</u>	<u>8758</u>	<u>2462</u>	<u>1247</u>

The test contains another similar row.

Test 6 — DIVISION

			At. _____
			Rt. _____
82)3854	43)1591	74)2664	31)1953

The test contains 5 other rows.

The following directions are printed on the first page of the test sheet :

There are more examples in each test than you can work out in the time that will be allowed. Answers do not count if they are wrong. Begin and stop promptly at signals from the teacher. Place the test in position on your desk so that you can open it quickly when the signal is given to begin, but do not open it until the signal is given.

After all of the tests have been completed have the pupils exchange papers. Read the answers aloud and have the children mark each example that is correct "C." Count the number of examples attempted and the number of "C's" and write the numbers in the proper spaces at the top of the tests. Examples partially completed or partially right are not counted.

Before collecting the papers have the records transcribed to the first page. The teacher should verify a sufficient number of records to make certain that the pupils have marked the papers and transcribed the results correctly.

Test	1	2	3	4	5	6
Number of examples attempted						
Number of examples right						

The Monroe Survey Tests. — These comprise :

1. Combinations in the four fundamental operations.
2. Single column addition.
3. Decades in subtraction.
4. Multiplication by a number of 1 order.
5. Short division.
6. Column addition of 5 addends — 4 figures each.
7. Multiplication by numbers of 2 orders.
8. Long division by numbers of 2 orders.
9. Column subtraction.
10. The four operations in fractions.
11. Correct placing of decimal point in division of decimals.

Monroe states that the purpose is “to provide a single score which will be a general measure of the pupil’s ability to perform the operations of arithmetic.” The total working time is $24\frac{1}{2}$ minutes.

The Woody Tests. — Time : 20 minutes for each of the four tests. The tests are :

1. Addition, subtraction, multiplication, division of integers.

2. The four operations in fractions and mixed numbers.
3. The four operations in decimals.
4. The four operations in denominate numbers.

The Woody tests are intended to measure the type and difficulty of examples that a class can solve correctly, and the examples in each test are arranged in order of progressive difficulty.

Too few examples are given in any one type to test a pupil's ability in that type. Only nine combinations in division occur, and only one example in quotient with remainder within the combinations. $\frac{7}{8}$ of 624 is given as a division example. The tests are unique in that almost every variety of example in each operation occurs in a single test.

The Monroe Reasoning Test. — This is a problem test containing forty-five simple and complex problems involving the four fundamental operations, simple measurements, and percentage. Credit is given for the correct principle or method used in solving a problem, and for the correct answer. The purpose of the test is "to find how well the pupils can understand the form of statement used in the problems." It is not a time test "but those who desire a rate score may secure it by requiring each pupil to draw a line around the number of the problem he is working on at the end of 10 minutes." The list of problems is a good one and the method of scoring is satisfactory.

The Stone Reasoning Test. — This test contains twelve problems increasing in difficulty, covering the fundamental operations in simple and complex problems in integers and easy fractions. The time allowed is 15

minutes. The method of scoring is quite intricate and measures the speed of the pupil's reasoning. The problems are marked right or wrong according as the reasoning is right or wrong regardless of whether the answer is correct or not. The accuracy of the reasoning may also be measured if desired. The problems are of the usual textbook variety.

The Starch Reasoning Test. — This test is similar in plan to the Woody scale for fundamentals. Fifteen problems of unequal difficulty are arranged in order of progressive difficulty. A pupil's score is the highest step done correctly. The list is perhaps a better one on the whole than that of the Stone Reasoning Test, but the problems are of the same general character.

The Buckingham Problem Test. — This test covers simple and complex problems, some to be solved mentally and some in written form, and involving the four fundamental operations, some fractional problems, some common measures, and a few problems in percentage. There are forty problems in all. Pupils are to be given all the time they can usefully employ, up to an hour and a quarter. No credit is given for any problem whose answer is numerically wrong. A pupil's score is the difficulty value of the hardest problem he answers correctly minus any failures on preceding problems. The score value is placed at the left of each problem.

The excerpt on the opposite page shows the first two and last problems of the test.

The Stevenson Problem Analysis. — Four leading questions are asked about each problem. Twelve problems are covered in all. 24 is the perfect score. It is

Value (42)	1. A baseball team took 12 players on a trip. The trip cost the team \$36. How much was that for each player?	ANSWER
(44)	2. An automobile was run 30 miles every day for a week. How many miles did it go?	ANSWER
(73)	15. Two boys agreed to cut a lawn for 90 cents. The first boy worked one hour and the second boy worked 4 hours. How much money should each receive?	ANSWER

not a speed test; ample time is to be allowed for all to finish.

Following are the directions given for the test and an illustration of one of the twelve problems:

Look at the problem below. Read it carefully, then read the part headed A. Four answers are given to the question, "Which of the following facts are given in the problem?" Only one of the answers is correct. Find the correct answer. If more than one answer seems correct, select the best one. The correct answer is number 1, "The total number of apples put in the sample boxes." Write the "1" in the parentheses to the left of the question.

Now read question B and its four answers. Which is the correct answer? Write the number "2" just to the left of question B. Read question C. Find the best answer and write its number just to the left of the question. Do the same for question D.

III. 720 fancy apples are put up in sample boxes of five apples each. How many boxes will it take to hold all the apples?

() A. WHICH OF THE FOLLOWING FACTS ARE GIVEN IN THE PROBLEM?

1. The total number of apples put in the sample boxes.
2. The cost of the apples per box.
3. The selling price of the apples.
4. The weight of each box of apples.

() B. WHICH OF THE FOLLOWING THINGS ARE YOU ASKED TO FIND OUT IN THE PROBLEM?

1. The number of bushels of apples put up in the boxes.
2. The number of boxes it will take to hold all the apples.
3. The number of apples put in each box.
4. The amount of money received from the sale of the apples.

() C. WHICH OF THE FOLLOWING IS THE MOST REASONABLE ANSWER?

- | | | | |
|------|-----|-----|----|
| 1. | 2. | 3. | 4. |
| 3600 | 360 | 144 | 72 |

() D. WHICH PROCESS SHOULD YOU USE IN SOLVING THE PROBLEM?

- | | |
|-------------|----------------|
| 1. | 3. |
| Addition | Multiplication |
| 2. | 4. |
| Subtraction | Division |

The Courtis Reasoning Tests. — The first test consists of 16 simple problems which are not to be worked. The pupils read the problem, decide on the operation that should be used, and write the sign of the operation. The second test consists of eight "two-step problems" which are to be worked in the order in which they occur. The scores are based upon the number of attempts and the number of rights. The first of these two problem tests is Test 6 of Courtis Research Tests, Series A, and the second test is Test 8 of the same series.

Test 6 does not yield accurate measures, owing to the fact that children who are very poor in reasoning but are given to guessing, are sure to get, on the average, about a quarter of their operations correct.

Test 8 requires not only correct reasoning, but also accurate computation. The score of reasoning is quite sure to be too low because it is cut down as a result of errors in computation.

The Otis Arithmetic Reasoning Test. — This test was used originally as "Test Five of Otis Group Intelligence Scale," a series of tests used in classifying the men drafted to serve in the World War. It has twenty problems, nine simple and eleven complex. The person examined has nothing to write except the numerical answer, for which a block space is provided. For anyone at all competent in computation no written work would be necessary, but those who feel the need of written computation are permitted to use it.

The time allowed is six minutes, and the score is the number of correct answers. A table is provided for converting the scores into "arithmetic ages." "To find

an arithmetic quotient, divide the pupil's arithmetic age by his chronological age when both are expressed in months." This quotient is an index of ability in arithmetic reasoning and shows rather high correlation with general intelligence.

The Schorling-Clark Practice Tests. — These tests are for use in reaching a definite standard in fundamental operations in integers, fractions, decimals, and per cents. They are arranged in book form and designed for daily practice testing. Each test takes three minutes. The pupil takes the first test not yet completed perfectly by him, marks his own work from the answers in the back of the book, and records the number of the trial made and whether it is right or wrong. These tests are convenient, their content is good, and they serve an excellent purpose — systematic daily practice to meet a definite standard.

The Fowlkes-Goff Practice Tests.¹ — This is an excellent set of ninety-five practice tests and five achievement tests for the seventh, eighth, and ninth years. As the authors themselves state, there is enough material to supply a daily practice test for half a term or a practice test every other day for a year.

The pupil may find his own grade on each test by dividing the number of examples correctly worked by the number attempted. This is an achievement grade only; the tests have not been standardized as to speed.

The content covers the four fundamental operations on integers, fractions, and decimals, arranged in a careful sequence. The combinations are fully covered, including the decade combinations in addition, products increased

¹ Published by Macmillan.

by an integer in multiplication, and quotients with remainders in division, within the tables. Much practice is given on column addition, including several pages of single columns. Multiplication by numbers of one and two orders, short and long division, all receive proper attention. The other work covered is: reductions of fractions to higher and lower terms and to decimals; addition, subtraction, and multiplication of single fractions and of mixed numbers; division of an integer by a fraction; the four fundamental processes in decimals. The achievement tests are carefully placed after a series of practice tests. The authors state that if a pupil cannot work all the examples in any such test, he needs to work the practice tests preceding that achievement test.

The tests are in good print, put up in convenient pads, and have proper spaces for a pupil's record. No key is provided for self-checking. Our one criticism is the direction given at the top of each addition page, "Always add downward." The teacher, however, can easily modify this direction to suit her own needs. A good modification might be "Always add both up and down."

Many other similar tests are in the market. Every teacher should read some good book on this subject.¹

Standard tests and the methods of administering them are still open to criticism.² It is to be expected that their advantages will be retained, their faults corrected, and their limitations understood.

¹ MONROE, W. S. — *Theory of Educational Measurements*; Houghton Mifflin Company. WILSON, G. M., and HOKE, K. J. — *How to Measure*; McCALL, W. A. — *How to Measure in Education*; KELLY, T. L. — *Statistical Method*; The Macmillan Company.

² UPTON, C. B. — "The Influence of Standardized Tests on the Curriculum in Arithmetic"; *The Mathematics Teacher*, April, 1925.

NON-STANDARD TESTS

Suggestions for making tests more beneficial to pupils.

— It will be noted that the content covered in the standard tests is limited. Much is still left for the non-standard tests to take care of, and this is as it should be. Unfortunately, the test lesson sometimes becomes a bugbear in the student's life. There are perhaps reasons why this form of lesson has come to be unduly dreaded. One is the fact that so much emphasis has been placed upon it as the basis of promotion, *of a failing or a passing mark in a course*. If a single test is to determine an important outcome, a dread of it may cause a nervous tension that inhibits normal mental reactions. A second reason may be a lack of sufficient preparation. Too long or too difficult a test discourages a pupil and sets up unpleasant associations for this type of lesson. A third reason why pupils dread tests is often their lack of comprehension of the purposes of a test. A pupil does not always understand that this form of recitation is as much a part of the learning process as is the development or drill lesson — that the teacher has as great a desire to help pupils overcome weaknesses in crystallizing their knowledge of a subject and expressing their understanding of it in written form as he has in making a development lesson clear or a drill lesson effective.

It may be well to inquire whether anything can be done to make the test period more pleasurable and withal more profitable. Let us consider some methods of handling the test lesson with the hope that these may prove helpful in bringing about such a state of affairs.

1. Arrange the work in such a way that the first trial on a test is optional. Announce that a test on such a piece of work will be given at such a time. Those who want to try it, may do so, but no one is compelled to take it. Those who succeed on first trial may do some extra work for extra credit. Those who do not try it will have the extra time for more study on the subject. Those who try it and fail will receive no penalty; they will have had the practice and can profit by their mistakes on the first trial when the second test on the same topic is given. If this plan is carried out over a series of weeks the teacher will find a growing tendency among his pupils to try the test the first time. Pupils like to feel the satisfaction of having attained the first time; they like to attempt a thing that does not have to be done under compulsion, and they like a chance to do the extra work. The extra work should always be systematically planned and should be entertaining and profitable. Puzzle problems and mathematical curiosities can be utilized for this purpose. Constructive work of various kinds may also be used. Whatever work is done at these times should be systematically kept on file in booklet or some other convenient form for credit. A teacher who tries this plan will find the whole atmosphere of the test period changed.

2. Give as a part of each recitation a short written test on the work of the preceding days. This may be done at the beginning of each recitation. If a person becomes accustomed to doing a thing regularly, he thinks but little about it and acquires a certain poise and confidence that comes in no other way. It is convenient to keep these daily tests in booklet form so that the pupil has his own

record before him and can see his mistakes. Sometimes the pupils can grade each other's work and sometimes each grades his own. Sometimes the teacher will collect and grade the papers.

3. Use competition. Two groups in the same class, or two classes, may carry on a contest for a set period, the losing group entertaining the winners in some way, if desired. Or the winners may receive recognition of some kind from the teachers. A trophy may pass back and forth in such a contest between classes with some such understanding as that the class who wins three consecutive five-week periods will hold it permanently. In such a contest the pupils usually plunge into the tests with a vim.

4. Make occasional use of a test for diagnostic purposes. Let the pupil understand that the test is to be given to show him his weaknesses and not for purposes of record.

5. Set a certain standard and ask pupils to see whether they can reach it. Some of the standard tests mentioned above may be used, or a teacher can make up a great many simple standards of his own, such as (a) For oral test: Read these numbers as fast as you can: 11, 13, 16, 7, 15. Now see whether you can read the answers to these combinations at about the same speed:

$$\begin{array}{cccccc} 5 & 7 & 8 & 4 & 8 \\ 6 & 6 & 8 & 3 & 7 \end{array}$$

(b) Write the numbers from 1 to 20 as fast as you can; now beginning with 1 count by 3 as far as 60 and see whether you can write these results in about the same time; 1, 4, 7, 10, 13, and so on. Use any one of the 44 counting series (p. 30) in this way. (c) Dictate some good exercises in finding per cents of numbers. When

the teacher taps, the pupils write the answer. The teacher says, "Ready : $83\frac{1}{3}\%$ of 18." He taps and the pupils write "15," etc. Almost any subject may be treated in this way.

6. Ask pupils to prepare the questions for a test. Divide the class into several groups, make each group responsible for a certain test, and ask all of the group to submit questions. A leader selects questions from this list, and conducts the test. Two of the group are selected to grade the papers under the teacher's guidance.

Suggestions for saving the teacher. — Not only may tests become a dread to pupils but they may prove a heavy tax on the strength of the teacher. Much of his energy, which should be expended in more useful ways, is given to poring over piles of test papers, and scrupulously weighing each part of an answer. Through much experience and through observation, the writers have learned that the following items, patiently persisted in, greatly lighten the teacher's load in the matter of tests :

1. Make the whole matter as impersonal as possible. At his best, a teacher will not always deduct the same amount for the same mistake, but a great aid in accomplishing this is to read through the answers for question 1 in all papers ; then all for question 2, and so on. In this way he can hold in mind more readily the amount deducted for a given error. If possible, mark papers without looking at the names of the students.

2. Never give an important test without answering the questions yourself and allowing your pupils from two to three times as long as it takes you. Almost any test will be revised by the time you have actually tried it out for yourself.

3. Plan the test in such a way that the answers will be short. A half hour in careful planning will save two or three hours in reading. This applies particularly to the higher grades. Tests which can be answered by completing some phrase, or by yes or no, or by true or false,¹ are being used quite generally, and are as much of a gain over reading pages of manuscript as is electricity over a kerosene lamp. Yet the teacher's gain is not the pupils' loss; they receive as much help as from a test in which copious writing and less thinking are required. Thus, instead of setting the question, "Discuss how to compute the commission when an agent is buying," the following may be given. "Fill in the blanks for these statements: If the gross cost and commission are given, to find rate of commission, we first find the —— and then divide the —— by the ——." Instead of a medley of answers, many of them incomplete, as he would receive from his first question, the teacher will have but three sets of words to read: "actual cost, commission, actual cost." The pupil will probably have to do more thinking and will need to have a clearer idea of the situation to answer the second question than the first.

4. Always return test papers carefully marked. A teacher should give shorter tests if necessary, but carefully mark them so that the pupils can find their mistakes. If the teacher makes a clerical mistake, he should always adjust it, but otherwise he should rarely change a mark. If he has carefully followed the suggestions in No. 1, the chances are his marking is fair. He

¹ WRIGHT, H. C. — "Some True-False Examinations for Use in General Mathematics"; *The Mathematics Teacher*, XVIII, No. 2.

has used his best judgment and his record should stand, unless he has overlooked something or has made a numerical error. He should always invite a discussion of mistakes made, and a personal conference when a pupil feels an error has been made. In such a conference it is wise to state clearly *why* the error is a serious one, if a large deduction has been made. Pupils often fail to distinguish errors on important matters from the lesser ones.

5. Endeavor to have pupils obtain the correct attitude towards marks. The teacher does not *give* marks; the pupils *earn* them and whatever they earn the teacher *pays*, keeping an account of scores. The average at the end of a given time is what the pupil himself has earned, not what a teacher gratuitously presents to him.

The teacher's responsibility. The foregoing tests set up standards which pupils in schools should reach. The teacher who would successfully lead and direct pupils to such levels of efficiency must needs be himself more adept. It is the common practice of institutions for the preparation of teachers to require that all prospective teachers shall, with their first or freshman year in school, acquire the ability and facility to meet eighth-grade standards in the fundamental operations. On his own initiative and for his own satisfaction the teacher should exceed these standards and extend this greater mastery to all of the operations and practices of arithmetic. Conscious of his own mastery of the skills and controls involved in arithmetic, the teacher may then devote his mental energy to the directing of the efforts of his pupils to master arithmetic.

EXERCISES

1. Prepare an oral test for each of the following, giving grade for which it is prepared:

- (a) Addition and subtraction of numbers of two orders.
- (b) The 44 counting series.
- (c) Multiplication of a number of two orders by a number of one order.
- (d) Squares of numbers to 25 and cubes to 9.
- (e) Products of any two numbers up to 100, outside the combinations.
- (f) Per cent equivalents of business fractions.
- (g) Problems in profit and loss, direct case.
- (h) Problems in profit and loss, all the cases.
- (i) Problems in commission, direct case.
- (j) Problems in commercial discount, one and two discounts.
- (k) Problems in simple interest, direct case.
- (l) Problems involving denominate numbers.
- (m) Problems involving computations with fractional parts of a dollar.

2. Prepare a written test, naming the grade for which it is prepared: (a) In addition. (b) In subtraction. (c) In multiplication. (d) In division. (e) In the operations in fractions. (f) In the operations in decimals. (g) Problems involving denominate numbers. (h) Problems involving the five fundamental operations.

3. Prepare a good set of problems suitable for a review test on each of the following: (a) Profit and loss. (b) Commercial discount. (c) Commission. (d) Simple interest. (e) Areas of rectangles. (f) Volumes of rectangular solids. (g) Promissory notes and bank checks.

4. The Public School Publishing Company, Bloomington, Ill., puts up in sample sets the following standard tests: Cleveland Survey Arithmetic, Wisconsin Inventory, Monroe Diagnostic Arithmetic, Stevenson Problem Analysis, Buckingham Scale for Problems, Monroe Survey Arithmetic, Monroe Reasoning Arithmetic. (a) Send for these tests and select one or more to be given. (b) Conduct the test, mark papers and make out record sheets according to directions.

5. Compare the above tests as to their adequacy in respect to the combinations in the four operations.

6. Compare the above tests for work in (a) column addition; (b) column subtraction; (c) column multiplication; (d) short division; (e) long division.

7. Discuss the relative merits of the problem tests cited above.

8. Prepare a test suitable as a standard in column addition, using all of the 45 combinations and 50 or more different decades.

9. Do the same for column subtraction, using all of the 90 combinations.

10. Do the same for column multiplication, using the 45 combinations.

11. Do the same for short division using the 90 combinations.

12. Explain how you would use standard tests to measure the results of your teaching. Select the test or tests you would use in a fourth-year class; in a fifth-year class; in a sixth-year class.

13. Suppose you have found a group of pupils in the sixth or a higher grade, who are very slow and inaccurate in column addition but whose work otherwise is satisfactory. What tests would you select or devise for the purpose of determining the precise nature of each pupil's weakness? What are the possible explanations for deficiency in column addition? How would you proceed in overcoming these difficulties?

THE APPRECIATION LESSON

Characteristics of the appreciation lesson. — The purposes of the appreciation lesson in mathematics are perhaps satisfied if the result is an intensified and pleasurable interest in the subject, and a wholesome respect for mathematical science and its diversified applications.

One can no more like a subject without knowing it than he can like a person without acquaintance. The first requisite then for appreciating arithmetic is to understand it. A plea has been made all the way through this

book for motivating and presenting the subject in such a way that it will have *real meaning* for the child. If something new is taught, let it be presented in such a way that the child can understand it. If a drill is planned, let it be so motivated and conducted that he will enjoy it and appreciate its purposes; if a test is given, let it be so conducted as to produce satisfaction and a desire for self-improvement. In this sense, then, every lesson in mathematics may have something of an appreciative value for the pupil. The teacher also should have a clear, firm grasp of the principles involved, and facility in executing processes. Such a teacher will find a lasting joy in leading children to reconstruct the science which men have through the centuries wrought out in their efforts to count, to measure, to control the forces of the universe, and to reason upon the relations therein disclosed.

Appreciation of the uses of mathematics. — An intelligent respect for the subject may be inculcated by occasional talks in which attention is called to the numerous applications of mathematics in the everyday affairs of life. The clerk behind the counter must be accurate in his computations. Bank clerks must be accurate to the cent before their work for the day is done. The big bridges that are built may collapse if the engineers make any mistake in their mathematical calculations. Mathematicians work out the tables by which the guns are set in the big battleships so as to have just the range desired. The tradesman, who cuts his stove pipe by a knowledge of mathematical curves, finds the elbow will fit precisely and that he has a neat piece of work on first trial. The astronomers can adjust their big telescopes so as to study

the wonders of the heavens by working out mathematical calculations accurately. Scholars pondering the laws of mathematical science give to the world new discoveries which later on are advantageously used by scientists and others in practical ways. The discovery of logarithms and of the calculus are familiar illustrations to the point.

Appreciation of the history of mathematics. — A growing interest in the subject is often aroused by a knowledge of its historical development. Much of this material may be put in attractive story form. This has been well done by D. E. Smith¹ for the fundamental operations, including counting and systems of notation. Becoming interested in this line of investigation through his little book, a small group of students² attempted as a project a similar story-form history of measures of length.

The material for the story which they produced was found in encyclopedias and histories of mathematics. It consisted of the names and meanings of measures of length used by the Persians, Hebrews, Egyptians, Greeks, Romans, and early English. The interest centered around some activity of a child using these measures, in which the setting was made as true to history as possible. In this way the cubit, plethron, palm, pous, fathom, span, and hand were explained.

Noteworthy expressions of appreciation. — Some appreciation of mathematics may also come from knowing what many of the great thinkers of the world have said about the subject. Moritz has collected in his volume

¹ *Number Stories of Long Ago.*

² A mathematics class specializing in Kindergarten methods in the New York Training School for Teachers.

over a thousand splendid quotations.¹ Here are a few samples :

Mathematics stands forth as that which unites, mediates between Man and Nature, inner and outer world, thought and perception, as no other subject does.

FROEBEL.

There is an astonishing imagination, even in the science of mathematics . . . we repeat, there was far more imagination in the head of Archimedes than in that of Homer.

VOLTAIRE.

If the wit be too dull, they (the mathematics) sharpen it; if too wandering, they fix it; if too inherent in the senses, they abstract it.

BACON.

Many arts there are which beautify the mind of man; of all other, none do more garnish and beautify it than these arts which are called Mathematical.

BILLINGSLEY.

The belief that mathematics, because it is abstract, because it is static and cold and gray, is detached from life, is a mistaken belief. . . . It is just the ideal handling of the problems of life, as sculpture may idealize a human figure, or as poetry or painting may idealize a figure or a scene.

KEYSER.

Mathematics . . . does furnish the power for deliberate thought and accurate statement, and to speak the truth is one of the most social qualities a person can possess. Gossip, flattery, slander, deceit, all spring from a slovenly mind that has not been trained in the power of truthful statement which is one of the highest utilities.

DUTTON.

Truth is the same thing to the understanding as music to the ear, and beauty to the eye. The pursuit of it does really as much gratify a natural faculty implanted in us by our wise Creator as the pleasure of our senses.

ARBUTHNOT.

Mathematical knowledge adds vigor to the mind, frees it from prejudice, credulity, and superstition.

ARBUTHNOT.

¹MORITZ, ROBERT EDOUARD. — *Memorabilia Mathematica*; The Macmillan Company.

The motive for the study of mathematics is insight into the nature of the universe. Stars and strata, heat and electricity, the laws and processes of becoming and being, incorporate mathematical truths. If language imitates the voice of the Creator, revealing His heart, mathematics discloses His intellect, repeating the story of how things came into being.

CHANCELLOR.

For the effective exposition of the value of mathematical thought those who are interested in its philosophic side will enjoy the essays of a well-known present-day mathematical philosopher. He regards mathematics as one of the "great forms of spiritual activity," "a creation of the intellect whose aim, ideal, and beatitude is harmony."¹

The highest aim in mathematical study is the pursuit of truth. The reward is knowledge of a science which is exact and enduring and the possession of an art which is one of the highest — the art of right thinking.

EXERCISES

1. Mention any topics in arithmetic from the study of which you may have derived as a child some special satisfaction. Explain what qualities of the work itself, or of the manner of presentation, gave the study its pleasurable tone.

2. Mention any work you may have had in arithmetic for which you had as a child a special aversion. How do you account for the feeling of aversion?

3. Make a list of the things connected with the study of arithmetic that, judging from the reports of your fellow students, are pleasurable to most pupils; also those that are disliked by most pupils.

4. Can you devise ways of presenting certain of these disagreeable topics so as to make them more pleasant?

5. Mention a few things you have learned recently about the *uses*

¹ KEYSER, C. J. — *The Human Worth of Rigorous Thinking*, pp. 23, 64; Columbia University Press, 1916.

of arithmetic that could be used in the classroom to stimulate in the children a better appreciation of the subject.

6. Children in the lower grades as a rule like to work abstract examples. Is this a matter of appreciation? Explain.

7. For each of the grades select two or three topics that would have a pronounced value in developing appreciation.

8. Review the sections on history and select items that would interest children. In what grade could each item be used to advantage?

9. "The grounds of dislike towards certain subjects¹ are often to be found in the wrong conceptions we have as to the methods of teaching them, and not in any fundamental incompatibility between the subjects and the child-mind." Cite an illustration of the truth of this statement.

¹ LEWIS, E. O. — "Popular and Unpopular School Subjects"; *Journal of Experimental Pedagogy*, II, p. 89, 1913.

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3, 4, 5 always a Δ

1. Common fraction

Proper
improper

.615 = closed decimal
.02 $\frac{2}{3}$ = circulating "

2. Decimal fraction

Refer to Every teachers Problems
Percentage

Geometric
problems

6 principles of fraction

181 Prob

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177 cases of fractions

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